Model of Digital Image Sensor for Optics Curriculum: A Mathematical Derivation

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Abstract: The model of digital image sensor is usually introduced without derivation, which makes it inconvenient to include this model in the courses of optics curriculum. In present research, the required mathematical derivation has been performed.

1. Summary

It is common knowledge that the content of educational programs should be regularly enriched in line with the progress made in scientific papers. However, when we are trying to do so, we often find that some pieces of information needed for bringing new knowledge into the classroom are missing.

After CCD arrays came into common use, various mathematical models of the sampling process performed by such sensors have been proposed [1-3]. However, the formulas of these models usually were introduced without mathematical derivation, which made it difficult to use them in academic courses. Moreover, some of the models were later criticized for being not completely correct [1-3].

In this era of digital optics, mathematical model of sampling performed by the image sensor should be included in the optics curriculum, but to do this in the right way, we at first need to build clear derivation of such a model. By doing so, we will not only maintain a scientific tradition, according to which the formulas should not be taken for granted and should be derived, but also will make sure that we are taking the correct model.

Previous attempts to perform such a derivation [4] resulted in a series of important findings, but did not achieve desired level of methodological clarity. The purpose of present research was to perform mathematical derivation of digital image sensor model, which will be suitable for inclusion in the courses of optics curriculum.

2. The concept of the model

When recording light intensity distribution formed in the digital image sensor plane, the sensor performs both spatial and temporal sampling. However, in most applications, such as digital holography, digital interferometry and others, only spatial sampling plays an important role, whereas temporal sampling can be neglected.

In mathematical modelling of spatial sampling performed by the sensor with pixels of rectangular shape, the following geometrical parameters are to be taken into account [4]:

- the sizes of the detector (Δx_d and Δy_d),
- the sizes of the whole pixel (Δx and Δy),
- the total numbers of pixels along the x and the y axes respectively (N and M).

Let $I(x,y)$ be continuous light intensity distribution formed in the sensor plane and $I[n,m]$ be two-dimensional array of numbers which we finally obtain from the sensor. For the convenience of theoretical analysis elements of numerical array $I[n,m]$ can be formally represented as values of some discrete spatial signal $I_S(x,y)$. By mathematical model of digital image sensor we mean an equation, which shows how discrete signal $I_S(x,y)$ can be obtained from continuous intensity distribution $I(x,y)$. For examples of applications of such models, please, see references [5,6].

3. Two key ideas needed for derivation of the model

The first thing that we need to do in order to derive the desired model is to choose an appropriate mathematical description for the discrete signal $I_S(x,y)$ representing the elements of the numerical array $I[n,m]$.

Basically, there are two methods for discrete signals description [7]. In the first method, the discrete signal is described as a product of the initial continuous signal and some sampling function. In the second method, the discrete signal is described as a sum of shifted impulses weighted by the elements of some numerical sequence or array. For modelling image sensor with non-zero pixel fill-factor, the second method has to be used [7], therefore

$$I_S(x,y) = \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} I[n,m] \cdot \delta(x - n\Delta x, y - m\Delta y). \tag{1}$$

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The second thing that we need to do for the derivation is to understand the physical meaning of the numbers $I[n,m]$ obtained from the image sensor.

One of the CCD inventors G.E. Smith emphasized in his Nobel Lecture that ‘the basic unit of information in the device was a discrete packet of charge and not the voltages and currents of circuit based devices’ [8]. This means that, in the ideal case, each element of the numerical array $I[n,m]$ will be directly proportional to the value of charge packet q_A accumulated in the corresponding pixel during the accumulation time Δt_A. Thus,

$$I[n,m] \propto q_A . \tag{2}$$

4. A brief outline of the model derivation

Equations (1) and (2) suggest the following scheme for deriving the sensor model: $I(x,y) \to q_A \to I[n,m] \to I_S(x,y)$. In this derivation, the following two restrictions will be applied. First, light intensity distribution $I(x,y)$ will be constant during charge packet accumulation time. Second, $I(x,y)$ will be formed by monochromatic light with frequency ν.

Assume that the charge packet q_A in the pixel is formed by N_e conduction electrons which have been released due to absorption of N_{ph} photons incident on the surface of the pixel’s detector. If η is the quantum efficiency of the detector, then

$$q_A = e \cdot N_e = e \cdot (\eta \cdot N_{ph}). \tag{3}$$

The number of photons N_{ph} can be expressed in terms of the total amount of energy ΔW incident upon the detector surface during accumulation time Δt_A. In turn, the energy ΔW is defined by the accumulation time Δt_A and the flux of light energy Φ through the surface of the detector. Thus,

$$N_{ph} = \frac{\Delta W}{h\nu} = \frac{(\Phi \cdot \Delta t_A)}{h\nu}. \tag{4}$$

The flux of light energy Φ through the surface of the detector can be expressed in terms of the light intensity distribution $I(x,y)$ formed in the sensor plane. If the center of the detector with numbers n, m is located at point $(n\Delta x, m\Delta y)$, and the sizes of the detector are Δx_d and Δy_d then

$$\Phi = \int_{n\Delta x - \frac{\Delta x_d}{2}}^{n\Delta x + \frac{\Delta x_d}{2}} \int_{m\Delta y - \frac{\Delta y_d}{2}}^{m\Delta y + \frac{\Delta y_d}{2}} I(x,y)\ dx\,dy = \ldots = \left[I(x,y) \otimes rect\left(\frac{x}{\Delta x_d}, \frac{y}{\Delta y_d} \right) \right](n\Delta x, m\Delta y). \tag{5}$$

Substituting equations in the order (1) ← (2) ← (3) ← (4) ← (5), we obtain the final formula of the model:

$$I_S(x,y) \propto \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} \left\{ \left[I(x,y) \otimes rect\left(\frac{x}{\Delta x_d}, \frac{y}{\Delta y_d} \right) \right](n\Delta x, m\Delta y) \cdot \delta(x - n\Delta x, y - m\Delta y) \right\}. \tag{6}$$

Since due to proportion (2) we cannot obtain exact equality in formula (6), the constant coefficient, which appears in (6) after all substitutions, also has been omitted. Using the properties of ‘comb’ and ‘rect’ functions, we can further transform formula (6) into another form which will coincide with the models used in [5,6].

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