From Default Probabilities to Credit Spreads: Credit Risk Models Explain Market Prices

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ABSTRACT

Credit risk models like Moody's KMV are now well established in the market and give bond managers reliable default probabilities for individual firms. Until now it has been hard to relate those probabilities to the actual credit spreads observed on the market for corporate bonds. Inspired by the existence of scaling laws in financial markets by [1] and [2] deviating from the Gaussian behavior, we develop a model that quantitatively links those default probabilities to credit spreads (market prices). The main input quantities to this study are merely industry yield data of different times to maturity and expected default frequencies (EDFs) of Moody's KMV.

The empirical results of this paper clearly indicate that the model can be used to calculate approximate credit spreads (market prices) from EDFs, independent of the time to maturity and the industry sector under consideration. Moreover, the model is effective in an out-of-sample setting, it produces consistent results on the European bond market where data are scarce and can be adequately used to approximate credit spreads on the corporate level.

Keywords: credit risk modeling; default risk; credit spread; expected default frequency; actual default probability and risk-neutral default probability; bond pricing

JEL classification: C15; C51; C52; C53; G12; G13

1. INTRODUCTION

Most securities are, in one way or the other, subject to credit risk: the uncertainty surrounding a firm's ability to meet its financial obligations. As a result, bonds issued by companies generally pay a spread over the default-free rate of a government bond, which must be related to the probability of default. In this paper we develop a model that relates credit spreads of different times to maturity to default probabilities or expected default frequencies (EDF) as estimated by Moody's KMV. Our model provides a closed-form solution and is suitable for empirical testing.

Building on the access to monthly yield and EDF data at an industry level on the U.S. bond market, we estimate market prices from EDFs for various industry sectors. Comparing the model outcomes with market-consistent credit spread data during the time horizon starting in November 1995 and ending in December 2004, we find highly consistent results independent of the time to maturity and the industry sector under consideration. Moreover, the model is reliable on both U.S. and European bond markets and performs well independent of the location.

A possible application of the model is to exploit the functional relation between EDFs and credit spreads (market prices) to infer credit spreads from EDFs on an industry and on a corporate level where no yields are yet available.

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Estimating credit spreads from actual default probabilities either empirically or purely mathematically has been rarely attempted, to our knowledge. The study of [3] proposes a reduced-form or intensity-based approach to estimate a relationship between actual and risk-neutral default probabilities. The author uses U.S. bond yield data and long-horizon default frequencies by credit ratings rather than EDFs. A similar study recently performed by [4] looks at the relationship between default probabilities and default risk premia estimated from credit default swap (CDS) market rates.
The paper of [5] contains an empirical analysis of the relationship between actual and risk-neutral default probabilities using structural models. A formal conversion or even establishing a model is left for future research according to the authors. Reference [6] establishes such a link by relying on the standard Merton (1974) model (c.f. [7]). Bohn's study is fundamental to Moody's KMV latest web-based tool *CreditEdge Plus*^{TM} which combines EDFs with a valuation framework returning fair values for bonds, loans and CDS by strongly relying on their huge proprietary database (c.f. [8]). We studied the foundations of this model. However, we could only partly test it empirically due to the lack of appropriate data required for estimating model parameters.

In our own model, we find that credit spreads exhibit a scaling law with respect to the time to maturity. Aside from being reliable for industry sector indices on both the U.S. and European bond markets, the model can also be used to calculate approximate credit spreads on the corporate level. We show in an out-of-sample analysis that credit spreads are well predicted for short forecasting periods less than three months, given up-to-date default probabilities. A Monte Carlo study of simulated credit ratings (distances to default) supports the model. The temporal changes of a firm's credit rating can be modeled as independent draws from a loggamma distribution.

We summarize the quality of modeling results with respect to actual credit spreads by a quality measure G based on squared errors, where $-\infty < G \leq 1.0$. Small differences between model outcomes and credit spreads are summarized by a value of G close to one, whereas large deviations are indicated by low and negative values. We find highly consistent results independent of the time to maturity and the industry sector, with G values larger than 0.85.

The organization of the paper is as follows. We start by explaining the EDF and bond yield data we use in this study and the way we aggregate them into sector and rating indices. Then we describe the transformation of credit spreads to risk-neutral default probabilities by relying on a simple cash flow valuation of credit-risky bonds. In Section 3, we establish the model relating credit spreads to EDFs. In Section 4, we illustrate the modeling results and propose different methodologies for empirically testing the model. Finally, we present our simulation study based on loggamma-distributed rating changes, providing evidence for the power law of credit spreads with respect to the time to maturity.

2. INPUT DATA AND RISK-NEUTRAL VALUATION

2.1. Bond Yield Data

The bond markets provide yield curves for zero-coupon bonds issued by companies belonging to different industries and rating classes. Aggregated yield-curves at an industry and rating class level are provided by the *Financial Market Curve Indices* (FMCI) database of Bloomberg. We focus on yields of industry sectors composed of corporate bonds denominated in U.S. Dollar, which are issued by U.S. based companies, and which are rated by Standard & Poor's (S&P). The time samples start in November 1995 and end in December 2004. Throughout the paper, we denote the monthly discretization of this time sample by t_i where $i = 1, \ldots, n$.
Monthly yield data, determined at month end for industries and default-free U.S. government bonds, are available for different times to maturity T_j where $j = 1, \ldots, m$. For our analysis we retrieve yield data of the following industries and S&P rating classes to which we refer to as *sector indices* throughout the remainder of this paper: *Utility A, Utility BBB, Media BBB, Bank A, Bank BBB, Broker & Dealer A, Finance AA, Finance A* and *Telephone A*. We denote the time series of default-risky or defaultable yields for an arbitrary sector index by $Y_{j,i}$ and the series of default-free yields by $\overline{Y}_{j,i}$. As everywhere in the paper, j indicates the time to maturity, and i is the time series index (from 1 to n).

2.2. EDF Data

The expected default frequency (EDF) constitutes a key input quantity to this study and describes the annual probability of default for firms with publicly traded equity during the forthcoming year. It is a well established quantity and widely accepted in the financial services industry and has become a standard measure of corporate credit risk. EDFs are the outcome of Moody's KMV model which establishes a functional relationship between an index called distance to default and the probability of default. For a description of the mapping between the distance to default and the EDF measure we refer to [9] and [10]. The EDF of a company varies over time, reflecting the changing economic prosperity of the firm or its industry sector. It has been shown by [11] that EDFs are a leading indicator of default and allow to predict a downgrading of a firm ahead of rating agencies decisions.

The series of sector index EDFs cannot be directly retrieved from Moody's KMV *CreditMonitor™* software and has to be constructed from individual company EDFs within the appropriate industry sector and rating class. In a first step we select all U.S. based companies contained in a particular industry sector under consideration (i.e. Utility). We assume that the chosen industry sector consists of N individual companies $C_1(t_i), \ldots, C_N(t_i)$ each with a credit rating grade $R'_1(t_i), \ldots, R'_N(t_i)$ at t_i. On the one hand, the number of firms changes over time due to startups, mergers and closings. On the other hand, credit rating grades may vary as time evolves. We denote the one-year EDF value of company k at t_i by $p'_k(t_i)$ where $k = 1, \ldots, N$ and N being random. At every point in time t_i CreditMonitor returns for each company $C_k(t_i)$ a rating $R'_k(t_i)$ and a one-year EDF value $p'_k(t_i)$.

We construct the time series of EDFs p_i for an arbitrary sector index by calculating the median* value at each t_i from company EDFs of the chosen industry sector $\mathcal{S}$ and of a particular rating class $\widetilde{R}$, where $\widetilde{R} = \{AA, A, BBB\}$ and its index set is denoted by h.

$$\begin{aligned} p_i &= \text{median}\,(t_i; \check{\mathbf{p}}(t_i))\,, \qquad i = 1, \ldots, n \\ \check{\mathbf{p}}(t_i) &= (\check{p}_1(t_i), \ldots, \check{p}_N(t_i)) \\ \check{p}_k(t_i) &= \Big(p'_k(t_i) |\, C_k(t_i) \in \mathcal{S}, R'_k(t_i) = \widetilde{R}_h\Big)\,, \qquad k = 1, \ldots, N, \qquad h = 1, \ldots, 3 \end{aligned} \tag{1}$$

In addition to sector indices we form rating indices by summarizing appropriate industry sector data of identical rating classes and construct a global index comprising all sector indices under consideration. The purpose of this construction is to provide estimates of model parameters which are used to make inference for the case of industry sectors and firms where data are scarce. At each t_i we compute the yield of the global index by aggregating the yields of individual sector indices $(1, \ldots, d)$ by

$$\begin{aligned} Y^G_{j,i} &= \text{median}\,(t_i; \mathbf{Y}(t_i, T_j))\,, \qquad i = 1, \ldots, n \\ \mathbf{Y}(t_i, T_j) &= (Y_1(t_i, T_j), \ldots, Y_d(t_i, T_j)) \end{aligned}$$

The series of EDFs for the global index is constructed in a similar way:

$$\begin{aligned} p^G_i &= \text{median}\,(t_i; \mathbf{p}(t_i))\,, \qquad i = 1, \ldots, n \\ \mathbf{p}(t_i) &= (p_1(t_i), \ldots, p_d(t_i)) \end{aligned}$$

2.3. From Cash Flows to Risk-Neutral Default Probabilities

Yield curves of bonds subject to default risk are usually provided for zero-coupon bonds. This fact simplifies the cash flow analysis since there are no intermediate payments and all the interests and the principal are realized at maturity. Suppose that we have invested in a default-free and a default-risky zero-coupon bond of maturity T_j and with a face value $\overline{F}$. At any point in time t_i we determine the present value of these bonds as the discounted cash flow. In the case of the default-risky bond we have to regard the final cash flow F at maturity T_j as uncertain and determine its value by the expected value

$$\mathbb{E}[F] = q_{j,i} R\overline{F} + (1 - q_{j,i})\overline{F}$$

*Unlike the mean, the median is a robust measure which is not affected by the noisy behavior of some outlier companies.

where $q_{j,i}$ denotes the risk-neutral default probability at time t_i for maturity T_j and R corresponds to the recovery rate (i.e. the percentage of the principal to be paid in case of a default).
The essence of risk-neutral pricing is that risky investments should offer the same expected return as a risk-free investment (c.f. [12]). Under the assumption of risk-neutrality the current market value of a default-risky bond at t_i (its face value discounted at its risk-adjusted discount rate) is equal to its expected value at maturity T_j discounted at the risk-free rate

$$\begin{aligned} \frac{\overline{F}}{\left(1+Y_{j,i}\right)^{T_j}} &= \frac{\mathbb{E}[F]}{\left(1+\overline{Y}_{j,i}\right)^{T_j}} \\ \left(1+Y_{j,i}\right)^{-T_j} &= \left[R+(1-R)\left(1-q_{j,i}\right)\right]\left(1+\overline{Y}_{j,i}\right)^{-T_j} \ . \end{aligned} \tag{2}$$

We infer a theoretical relationship between the default-free yield, $\overline{Y}_{j,i}$, the default-risky yield, $Y_{j,i}$, and the risk-neutral default probability from Eq. (2) as follows:

$$q_{j,i} = \frac{1}{1-R}\left[1-\left(\frac{1+Y_{j,i}}{1+\overline{Y}_{j,i}}\right)^{-T_j}\right] \tag{3}$$

The risk-neutral default probability is uniquely determined only if we know the recovery rate R. Based on empirical results found in [13], [14], [15], [16] and [17], we assume a generic value of $R = 40\%$. We make the simplistic assumption that R stays constant over the considered time horizon and is independent of the choice of time to maturity. Further we assume the *same* recovery rate for all sector and rating indices.
We emphasize that the time series $q_{j,i}$ corresponds to the probabilities of default for the *entire remaining* time to maturity T_j and exceeds the annual quantity with increasing maturities larger than one year. We base later comparisons on an annual level and annualize the series $q_{j,i}$ as follows: Assume that for a time to maturity of one year a default-risky bond in survival pays the amount $(1-\tilde{q}_{1,i})F$. In case of a maturity of two years the survival probability is $(1-\tilde{q}_{2,i})^{T_2}$ and the survival cash flow amounts to $(1-\tilde{q}_{2,i})^{T_2}F$ and so on. The series of annualized risk-neutral default probabilities follows from:

$$\tilde{q}_{j,i} = 1-\left(1-q_{j,i}\right)^{\frac{1}{T_j}} \tag{4}$$

where $\tilde{\ }$ simply indicates *annualized* quantities.
From Eq. (2) we can further deduce the essential relationship between credit spreads, the probability of default, the recovery rate, and the default-free yield

$$\begin{aligned} s_{j,i} &= Y_{j,i}-\overline{Y}_{j,i} \geq 0 \\ &= \frac{1+\overline{Y}_{j,i}}{\left[R+(1-R)\left(1-q_{j,i}\right)\right]^{\frac{1}{T_j}}}-1-\overline{Y}_{j,i} \ . \end{aligned} \tag{5}$$

3. MODELING APPROACHES

Default-risky industry or corporate yields are often only partially available, or not at all. Thus a credit spread cannot always be easily inferred. It is the main goal of this paper to estimate credit spreads (market prices) from default probabilities proxied by EDFs. The following sections establish modeling approaches to estimate the series of risk-neutral default probabilities $q_{j,i}$ in Eq. (5) for different times to maturity from one-year EDFs.

3.1. The Brownian Motion Model

Credit risk models found in the finance literature are either based on a structural framework or on a reduced-form setting. Structural models rely on a contingent-claims approach to valuing corporate debt using the option pricing theory as proposed by Black, Scholes and Merton. The latter group of models is based on credit rating migrations and historical credit rating transition probabilities and assumes that the event of default is generated by some exogenous hazard rate process. Often firm-specific data relevant for structural models and for estimating underlying model parameters are not easily or not at all accessible to regular financial services companies. A

major problem of reduced-form models is the availability of realistic and regularly updated rating transition probability matrices. In the following we propose a continuous-time model of rating fluctuations that incorporates elements of both approaches.
We model a firm's credit rating by the distance to default and assume the following:

(i) The creditworthiness of a firm is modeled by its distance to default $X = (X_t)_{0 \leq t \leq T}$ which is assumed to fluctuate over time as a Brownian motion.

(ii) There exists a minimum boundary which corresponds to a default level d. It serves as an absorbing barrier and prevents the process X from recovering once it hits that level. For convenience this default level is defined to be at zero ($d = 0$), implying that solvent companies have a strictly positive distance to default, $X_t > 0 \; \forall t$.

(iii) The rating process X is assumed to neglect a drift term. - This is a simplistic assumption. The drift is rather difficult to estimate accurately and adding this term to the model provides almost no additional information.

(iv) The process X is assumed to start above the default level at $X_0 = x_0$ where $x_0 > 0$.

These assumptions are strong enough to arrive at a complete model of default probabilities. We consider a time interval $[0, T]$ where T corresponds to the time to maturity. We fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which there is a standard Brownian motion $W = (W_t)_{0 \leq t \leq T}$ to represent uncertainty. The actual or physical probability measure is denoted by $\mathbb{P}$. The information set generated by this Brownian motion up to and including time t is represented by the filtration $\mathbb{F} = \{\mathcal{F}_t \subset \mathcal{F} | \, t \in [0, T]\}$.
Let the process X follow a standard Brownian motion W starting at $X_0 = x_0$. That is,

$$X_t := x_0 + \sigma_X W_t$$

where $x_0 > 0$ and $\sigma_X > 0$ corresponds to the volatility of the rating process X. So far, we have not provided a scale of the process X, we only postulate its existence.
No default requires not only that the value of the rating process X exceeds the default level at maturity T but also demands that its running-minimum over time never hits the default barrier d. We define the probability of default $p(T)$ at maturity T by

$$p(T) = \mathbb{P}\left(X_T \leq d, \min_{0 \leq t \leq T} X_t \leq d\right) \qquad \forall t \in [0, T] \tag{6}$$

The evaluation of the joint probability in Eq. (6) is based on general computations and results of a stochastic process minimum hitting a lower boundary by relying on the strong Markov property and on the reflection principle. For a formal derivation and results we refer to [18], [19] and [20]. The probability of hitting the default barrier within the time interval $[0, T]$ starting now is

$$p(T) = 2\left[1 - \Phi\left(\frac{x_0}{\sigma_X \sqrt{T}}\right)\right] \tag{7}$$

where $\Phi(\cdot)$ denotes the cumulative of the standard normal distribution. The quantity x_0 can be regarded as the definition of the credit rating variable or more precisely its current value and is obtained by inverting Eq. (7).

$$x_0 = \sigma_X \sqrt{T} \Phi^{-1}\left(1 - \frac{p(T)}{2}\right) \tag{8}$$

where $\Phi^{-1}(\cdot)$ corresponds to the inverse of the cumulative standard normal distribution. From Eqs. (7) and (8) we establish a scaling law that directly relates default probabilities of different times to maturity. We reformulate

Eq. (7) for an arbitrary time to maturity T_j and substitute the result of Eq. (8) using a maturity of one year T_1 leading to

$$\begin{aligned} p(T_j) &= 2\left[1 - \Phi\left[\sqrt{\frac{T_1}{T_j}}\Phi^{-1}\left(1 - \frac{p(T_1)}{2}\right)\right]\right] \\ &= 2\Phi\left[\sqrt{\frac{T_1}{T_j}}\Phi^{-1}\left(\frac{p(T_1)}{2}\right)\right] \qquad j = 1, \ldots, m \end{aligned} \tag{9}$$

We observe that Eq. (9) is independent of the standard deviation of the rating process σ_X and the initial credit rating x_0. The probability $p(T_j)$ is not an annual default probability but describes the probability of default for the *entire* remaining time to maturity T_j. It can be annualized in the same way as shown in Eq. (4).
We estimate the series of risk-neutral default probabilities $q_{j,i}$ of Subsection 2.3 by

$$q_{j,i} = 2\Phi\left[\sqrt{\frac{T_1}{T_j}}\Phi^{-1}\left(\frac{p_i}{2}\right)\right] \tag{10}$$

where p_i corresponds to the series of one-year EDF values as determined in Subsection 2.2. We simply rely on empirical results and do not perform a proper mathematical change of measure (i.e. from a physical to a risk-neutral probability measure). Noticing this fact we do not expect Eq. (10) to accurately describe risk-neutral default probabilities from EDFs. Nevertheless, we attempt to approximate the credit spread by an *EDF implied spread* (EIS) by substituting these estimates of risk-neutral default probabilities in Eq. (5).

3.2. The Power Law Brownian Motion Model

A deficiency of the Brownian motion approach seems to be the lacking possibility of sudden credit rating losses. Empirical results (c.f. Fig. 1) show that the Brownian motion model does not describe reality well. It often appears that a firm's credit rating may suddenly deteriorate rapidly - in the worst case even leading to an immediate default. Contrariwise, multi-step upgrades are not likely to be observed in reality. We also assume that with increasing times to maturity there is a tendency of being downgraded rather than of being upgraded. The Brownian motion model has a survivorship bias meaning that a firm's credit rating stays too close to its present rating grade. Thus, we can no longer describe severe movements of the credit rating by a diffusion model of Gaussian type. To account for the possibility of sudden downgrades and the asymmetry in the credit rating we consider a power law. In Section 5, the link between the power law introduced here and the behavior of credit rating changes will be established.
We base our extended version of the Brownian motion model on empirical results rather than on a proper mathematical framework. In other empirical studies of financial asset price dynamics, it has been shown that scaling laws deviate from those expected from a Gaussian distribution (c.f. [21] and [2]). Inspired by this, we simply introduce additional parameters in Eq. (10) leading to

$$\tilde{q}_{j,i} = 2\Phi\left[c_i\left(\frac{T_1}{T_j}\right)^{\alpha_i}\Phi^{-1}\left(\frac{p_i}{2}\right)\right] \tag{11}$$

where $0 < \alpha_i < 1.0$ and $c_i \in \mathbb{R}$ are parameters estimated at every t_i. We emphasize that Eq. (11) is a relationship for directly approximating *annualized* risk-neutral default probabilities.[†]
The exponent α_i describes the empirical behavior of firms in the market. It mainly captures the overall movement and accounts for the scaling law of default probabilities with respect to the time to maturity. A small α_i characterizes a behavior where sudden credit rating losses are more important than gradual drifts. The other

[†]We used the same structure of the model to estimate *non-annualized* risk-neutral default probabilities but annualized them in the sequel by using Eq. (4). There is no parameter set for this alternative model that describes risk-neutral default probabilities well across all times to maturity. In the model we calibrated, risk-neutral default probabilities were underestimated for short and overestimated for medium and long times to maturity.
Notice that this non-annualized model version embeds the Brownian motion model if the we set $\alpha_i = \frac{1}{2}$ and $c_i = 1\ \forall i$.

parameter c_i describes the overall level of expected default probabilities and could be regarded as a risk premium or a market price of credit risk. We estimate α_i and c_i at each point in time t_i by simple linear regressions across all times to maturity T_j as follows:

$$\ln\left[\frac{\Phi^{-1}\left(\frac{\tilde{q}_{j,i}}{2}\right)}{\Phi^{-1}\left(\frac{p_i}{2}\right)}\right] = \ln c_i + \alpha_i \ln\left(\frac{T_1}{T_j}\right) + \varepsilon_{j,i} \qquad \forall i = 1, \dots, n \tag{12}$$

where $\ln\left(\frac{T_1}{T_j}\right)$ is used as the regressor variable and the dependent variable is composed of *annualized* risk-neutral default probability values $\tilde{q}_{j,i}$ and EDFs p_i. To estimate α_i and c_i for a specific sector index we assume that risk-neutral default probabilities can be computed from available yields of a related sector index or from aggregated global index yield data. Here the $\varepsilon_{j,i}$ are independent random variables with $\mathbb{E}(\varepsilon_{j,i}) = 0$ and $\mathrm{Var}(\varepsilon_{j,i}) = \sigma^2_{\varepsilon_i}$. We insert the estimates of α_i and c_i in Eq. (11) and directly infer the EDF implied spread from

$$s_{j,i} = \frac{1 + \overline{Y}_{j,i}}{\left[R + (1-R)\left(1 - \tilde{q}_{j,i}\right)^{T_j}\right]^{\frac{1}{T_j}}} - 1 - \overline{Y}_{j,i} \tag{13}$$

4. RESULTS AND MODEL TESTING

We look at the descriptive power of the individual models and compare EDF implied spreads to market credit spreads for sector indices where yield data is available. We summarize our findings by presenting the results for the global index. In addition to graphical visualizations we propose a statistic to assess the "fit" and to make inference about the quality of a model with respect to actual credit spreads. We partly adopt the idea of the r-squared statistic used to quantify a fit in the analysis of variance and regression.

Definition 4.0.1. *Quality of a model. Let* $n \in \mathbb{N}$, $\mathbf{Z} = (Z_1, \dots, Z_n)$ *be a random vector with realizations* $\mathbf{z} = (z_1, \dots, z_n) \in \mathbb{R}^n$. *Denote the estimator of* $\mathbf{z}$ *by* $\hat{\mathbf{z}} = (\hat{z}_1, \dots, \hat{z}_n) \in \mathbb{R}^n$. *Define the statistic* G *for the goodness of a model by*

$$\begin{aligned} G &:= 1 - \frac{\sum_{i=1}^{n} (z_i - \hat{z}_i)^2}{\sum_{i=1}^{n} (z_i - \bar{z})^2} \qquad -\infty < G \leq 1 \qquad \text{where} \\ \bar{z} &= \frac{1}{n}\sum_{i=1}^{n} z_i \end{aligned}$$

This statistic results in values close to one if differences between credit spreads and their model approximations are small. Larger deviations are summarized in smaller and negative values for G.
As a representative example we plot modeling results for the time horizon from November 1995 until December 2004 for the global index.

We observe in Fig. 1 that market consistent credit spreads are overestimated by the Brownian Motion model for the whole time horizon under consideration. Including additional parameters, possibly accounting for a risk premium, a market price of credit risk or the market mood, leads to superior "fits" as indicated by the results of the Power Law Brownian Motion model and a value of $G = 0.97$. We recognize that this model seems to capture strong increases of credit spreads better than sudden downfalls. We observe that the PLBM model produces adequate results independent of the state of the economy. For instance the economic downturn emanating from the collapse of the "dot com bubble" which led to a large number of bankruptcies is remarkably well explained by the PLBM model.

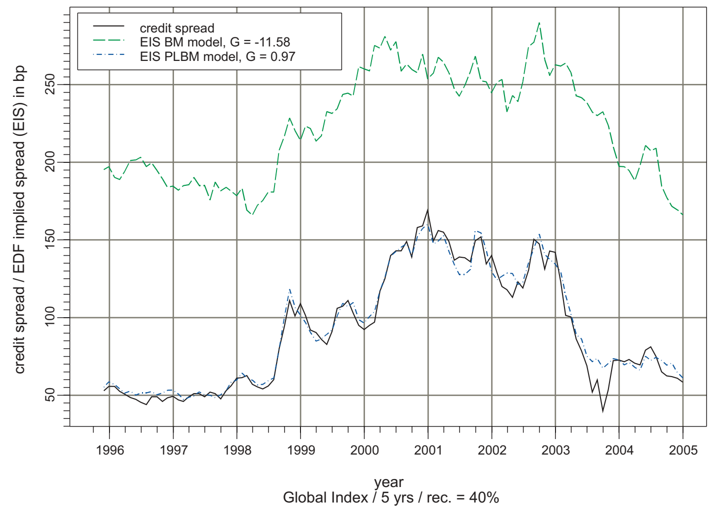

Figure 1. Comparison of model results for the global index for a time to maturity of five years. The solid curve represents market credit spreads. Results of the BM and the PLBM model are displayed by the dashed and the dashed-dotted curves, respectively.

We show in Fig. 2 the behavior of annualized risk-neutral default probabilities in dependence of the time to maturity for an average month within the time sample November 1995-December 2004.

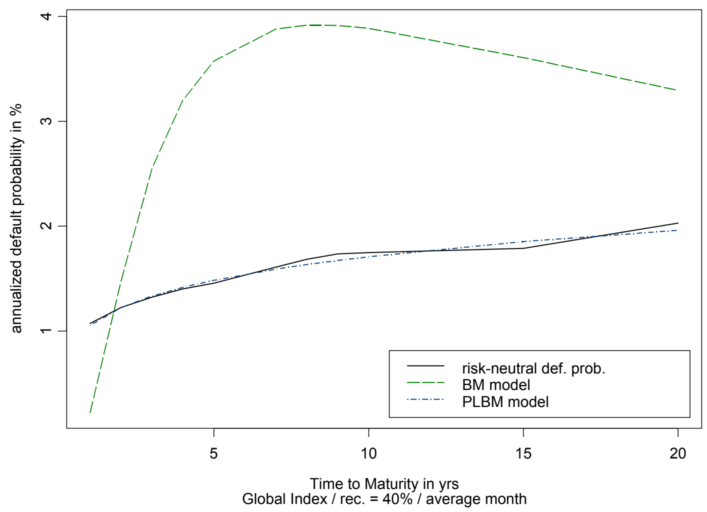

Figure 2. Annualized default probabilities for the global index in dependence of the time to maturity for an average month during November 1995 and December 2004. The solid curve represents annualized risk-neutral default probabilities. Results of the BM model are displayed by the dashed curve and the ones of the PLBM model are shown by the dashed-dotted line.

Figure 2 indicates that the BM model underestimates annualized risk-neutral default probabilities for short and overestimates them for medium and long times to maturity. Reasonable approximations of that model are merely found for a time to maturity of two years. Contrariwise, we are able to quite accurately explain annualized risk-neutral default probabilities with our model for *all* times to maturity under consideration. Due to the annualization we can directly compare default probabilities of different times to maturity and find, as expected, that *annualized* default probabilities increase with longer times to maturity.

4.1. Out-of-Sample Test

Our model explains most of current credit spreads. The aim of this subsection is to examine the performance and to show the limits of the model in an out-of-sample setting. We divide our time sample (November 1995-December 2004) into an *in-sample* and an *out-of-sample* period. In the in-sample period starting in November 1995 and ending in November 1998 we assume that yield data and EDF values are available. The out-of-sample period is defined as the time interval from November 1998 until December 2004. At each point in time in the out-of-sample period we perform model forecasts for one month to twelve months in the future conditioned on the information available as indicated in Fig. 3.

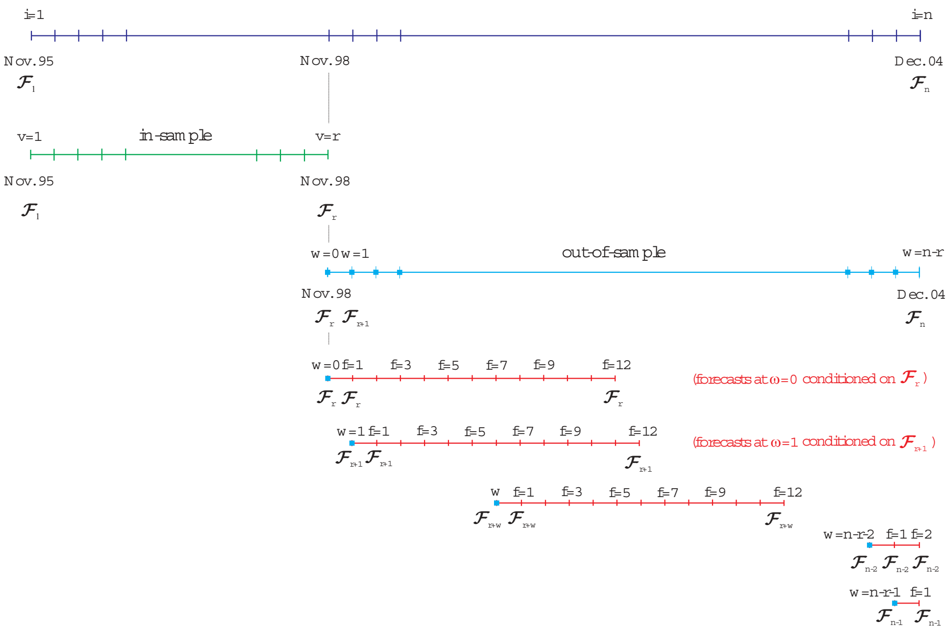

Figure 3. Time samples considered for the out-of-sample testing of the PLBM model.

We denote the index set of the in-sample time points t_v by $v = 1, \ldots, r$ and refer to the appropriate information set up to and including time r by the filtration $\{\mathcal{F}_r : r \geq 1\}$, where $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \ldots \subseteq \mathcal{F}_r$. The accumulating information set of the out-of-sample observations t_w, where $w = 0, \ldots, n-r$, is represented by the filtration $\{\mathcal{F}_{r+w} : w \geq 0\}$, where $\mathcal{F}_r \subseteq \mathcal{F}_{r+1} \subseteq \ldots \subseteq \mathcal{F}_n$. For model forecasts within the out-of-sample period we assume that we have complete information on EDF values, however, only partial information of yield data which gradually becomes known as time evolves. For this out-of-sample valuation we estimate the parameter series α_i and c_i from appropriate time series models. Applying standard techniques of time series analysis reveal a mean-reverting behavior for both parameter series. Based on model selection criteria such as Akaike (AIC) and Schwarz-Bayesian (BIC) we describe this behavior by an autoregressive type of model. For better readability and to avoid cumbersome notation we denote the index set of the time for this out-of-sample testing by the subscript $u = r + w + f$.

$$\alpha_u = \alpha_{u-1} + \nu_w \left(\bar{\alpha}_w - \alpha_{u-1}\right) + \phi_w \left(\alpha_{u-1} - \alpha_{u-2}\right) + \varepsilon_u^\alpha \quad (14)$$

$$c_u = c_{u-1} + \eta_w \left(\bar{c}_w - c_{u-1}\right) + \psi_w \left(c_{u-1} - c_{u-2}\right) + \varepsilon_u^c \quad (15)$$

where

$$\bar{\alpha}_w = \frac{1}{r+w}\sum_{i=1}^{r+w}\alpha_i$$

$$\bar{c}_w = \frac{1}{r+w}\sum_{i=1}^{r+w}c_i$$

As f-step predictions (f referring to the length of the forecasting period) we use the expectations $\mathbb{E}[\alpha_u|\mathcal{F}_{r+w}]$ and $\mathbb{E}[c_u|\mathcal{F}_{r+w}]$, conditioned on the information available up to the appropriate point in time in the out-of-sample. Since it is rather difficult to arrive at a direct formula for an f-step prediction we use the idea that for $f \geq 1$ the predictions $\mathbb{E}[\alpha_u|\mathcal{F}_{r+w}]$ and $\mathbb{E}[c_u|\mathcal{F}_{r+w}]$ are evaluated recursively by $\mathbb{E}[\alpha_{u-1}|\mathcal{F}_{r+w}]$ and $\mathbb{E}[c_{u-1}|\mathcal{F}_{r+w}]$, respectively. We further assume that the innovations ε_u^α and ε_u^c have the martingale difference property with respect to $\mathcal{F}_{r+w}$, meaning that $\mathbb{E}[\varepsilon_u^\alpha|\mathcal{F}_{r+w}] = 0$ and $\mathbb{E}[\varepsilon_u^c|\mathcal{F}_{r+w}] = 0$. We estimate the underlying parameters of the time series models, ν_w, ϕ_w, η_w and ψ_w by multiple linear regressions and obtain the f-step forecasted annualized risk-neutral default probability at t_w by

$$\tilde{q}_{j,u} = 2\Phi\left[c_u\left(\frac{T_1}{T_j}\right)^{\alpha_u}\Phi^{-1}\left(\frac{p_w}{2}\right)\right]$$

We plot the results of this out-of-sample test for the global index and a fixed time to maturity of twenty years.

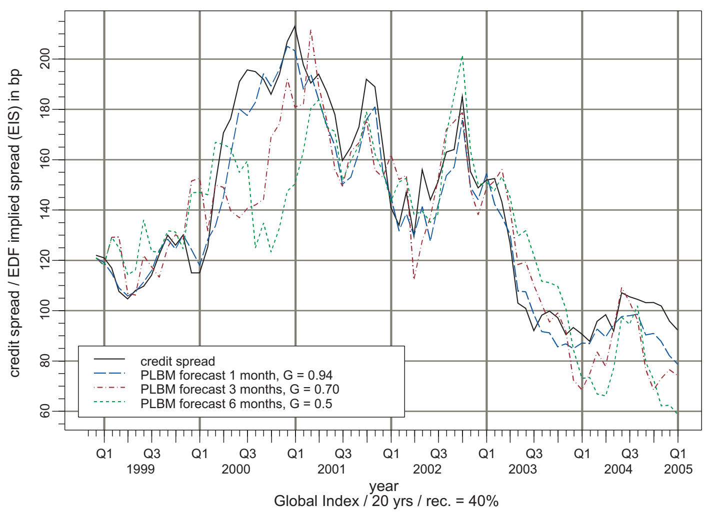

Figure 4. Model forecasts of credit spreads for the global index for different lengths of forecasting periods. The series of market credit spreads is shown by the solid curve. PLBM forecasts for one month, three and six months are represented by the long-dashed, dashed-dotted and short-dashed lines, respectively.

We find that credit spreads are quite well forecasted and summarized with relatively high values for the G statistic for short forecasting periods. Independent of the time to maturity under consideration we observe a similar quality of forecasting results but clearly detect larger deviations from market credit spreads for longer forecasting periods. This methodology allows to determine current credit spreads for an arbitrary sector index based on previous yield data, current EDFs and parameter values estimated from the information available.

4.2. Testing on the European Bond Market

This section serves the purpose of analyzing whether the model is independent of the location and can be used to explain credit spreads on other bond markets as well. We entirely rely on the theoretical framework proposed in Sections 2 and 3 and present the results for the *European Utility A* sector index.

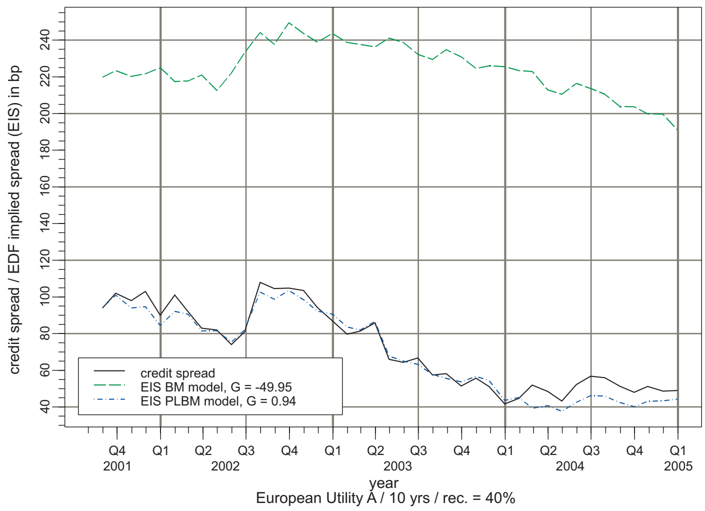

Figure 5. Market credit spreads (solid curve) and its approximations by EDF implied spreads (EIS) for the A-rated European Utility sector index and a time to maturity of ten years. The results of the BM and the PLBM model are represented by the dashed and dashed-dotted lines, respectively.

We find that on average less yield data on other bond markets is available from the FMCI database in the time sample under consideration. As in previous figures we observe again that the EIS determined from the BM model strongly overestimates the credit spread resulting in a low value of G (i.e. $G = -49.95$). Remarkable and consistent modeling results are achieved with the PLBM model leading to a value close to one for the G statistic (i.e. $G = 0.941$). This practical example provides some evidence that the PLBM model is also reliable on other bond markets as well.

4.3. Testing on the Corporate Level

While EDF values can be obtained for almost all companies with publicly traded equity, it often appears that yield data for individual firms is hardly accessible and a corporate credit spread cannot be inferred. We test the PLBM model for some companies of different industry sectors providing enough yield data within the initial time sample under consideration. Once a corporate bond is publicly issued its time to maturity shortens as time evolves. Instead of assuming a fixed time to maturity T_j we have to reformulate the key Eqs. (3), (5), (11) and (13) in dependence of a decreasing time to maturity δ_i

$$\begin{aligned} \boldsymbol{\delta} &= (\delta_1, \ldots, \delta_n) \\ \delta_i &:= \overline{T} - t_i \qquad i = 1, \ldots, n \end{aligned}$$

where $\overline{T}$ denotes the maturity date for an arbitrary corporate bond.
We illustrate the modeling results for a bond maturing in May 2003 of the American aircraft and aerospace manufacturer Boeing.

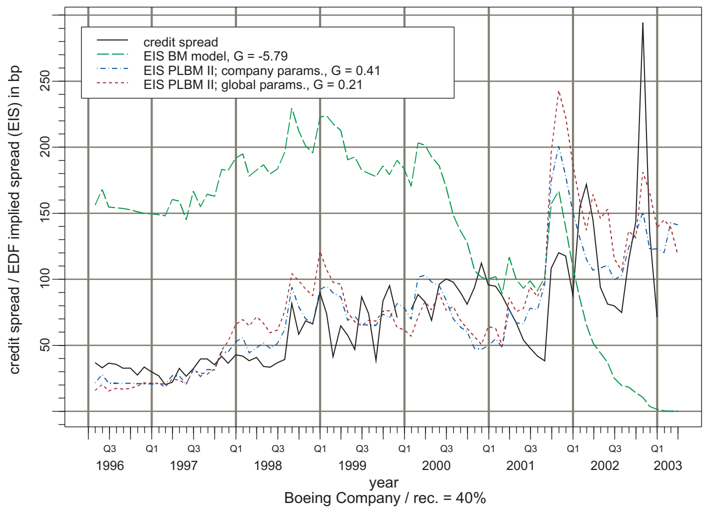

Figure 6. Market credit spreads (solid curve) and its approximations by EDF implied spreads (EIS) for the Boeing company's corporate bond maturing in May 2003. The results of the BM and the PLBM model using company specific estimates for α_i and c_i are represented by the dashed and dashed-dotted lines, respectively. The short-dashed curve displays PLBM model results using company specific EDFs but parameter estimates determined from global index yield and EDF data.

The decreasing behavior of BM modeling results towards the bond's maturity is caused by the diminishing time to maturity. The PLBM model using parameters α_i and c_i estimated from company specific data seems to describe credit spreads reasonably well at the beginning of the time sample but reveals larger deviations with the start of the year 1999. For the hypothetical case where only company EDFs but no appropriate yield data were available, α_i and c_i have to be estimated from global index data. We see that the PLBM model may be used to describe the overall movement and to provide a first guess of corporate credit spreads but clearly reaches its limits reflected by a relatively low value for the G statistic (i.e. $G = 0.41$).

5. SIMULATION STUDY

The power law behavior of the credit rating and the PLBM model itself are based on heuristic arguments and have not yet been supported by a proper mathematical framework. Instead of entering stochastic analysis we want to verify and provide evidence for the modeling results by relying on an independent Monte Carlo simulation study. The aim is to provide a qualitative description of the process of credit rating dynamics at the origin of the scaling law of time to maturity for annualized default probabilities.

5.1. Simulation Model

We consider the ratings of d firms in a population, count the number of surviving companies and infer an annualized default rate.

We model the creditworthiness of firm l, where $l = 1, \ldots, d$, by its distance to default and assume the default barrier to be set at 0. For each time to maturity T_j, where $j = 1, \ldots, m$, we consider a d-dimensional random vector of distances to default $\mathbf{D}(T_j) = (D_1(T_j), \ldots, D_d(T_j))$. We further assume that all companies have the same initial credit rating explained by an identical initial distance to default, $\mathbf{D}(T_0) = \mathbf{d}(T_0) = (d_1(T_0), \ldots, d_d(T_0))$, $d_l(T_0) = d(T_0)$ and $d(T_0) \in \mathbb{R}^+ \ \forall l$, and that a solvent company has a positive distance to default, $D_l(T_j) > 0$. Note that we look at the same d companies across all times to maturity T_j and observe a "dying off" of firms with T_j increasing. Startups of new companies as the time to maturity evolves are not taken into account in this simulation model, because these new companies do not yet have corporate bonds at simulation start.

The change of the credit rating in one time step is modeled by a loggamma distribution with a heavy tail on the downside. This function is asymmetric with a shift parameter which can model an overall downward (or upward) trend. For a definition of the loggamma density function we refer to [22]. We determine the distance to default for an arbitrary company l as follows:

$$
\begin{aligned}
\mathbf{X}(T_j) &= (X_1(T_j),\ldots,X_d(T_j)) \qquad j=1,\ldots,m \\
\mathbf{Z}(T_j) &= (Z_1(T_j),\ldots,Z_d(T_j)) \qquad j=1,\ldots,m \\
D_l(T_j) &= \begin{cases} \max\left(D_l(T_{j-1})-X_l(T_j),0\right) & : \text{ if } D_l(T_{j-1})>0 \\ 0 & : \text{ if } D_l(T_{j-1})=0 \end{cases}
\end{aligned} \tag{16}
$$

and

$$
X_l(T_j) = a e^{Z_l(T_j)} - b \qquad Z_l(T_j)>0, a>0, b\geq 0 \tag{17}
$$

where $D_l(T_j)\in\mathbb{R}_0^+$, $Z_l(T_j)\sim\Gamma(\alpha,\beta)$, $\alpha>0$ and $\beta>0$ and $D_l(T_j)$ is evaluated recursively from $D_l(T_{j-1})$. We explain the uncertainty of a firm's creditworthiness by the random variable $X_l(T_j)$ which is loggamma distributed, $X_l(T_j)\sim\mathrm{LG}(\alpha,\beta)$, with tail parameter α and scale parameter β. The parameter a in Eq. (17) is simply a scaling factor which enables to measure the distance to default in arbitrary units and b represents a shift along the x-axis allowing for individual upgrades. Such upgrades are reflected by a moderate increase of the distance to default and credit rating grade, an event that often occurs in reality.

We map the distance to default to probabilities of default by counting the number of surviving companies and by inferring an annualized default rate. Among the population d, $k(T_j)$ companies survive one year later, where $k(T_j)\leq d$. For each time to maturity T_j we count the number of non-defaulting companies with a strictly positive distance to default

$$
\begin{aligned}
\mathbf{k} &= (k(T_1),\ldots,k(T_m)) \\
k(T_j) &= \sum_{l=1}^{d} \mathbb{1}_{\{D_l(T_j)>0\}} \qquad j=1,\ldots,m
\end{aligned}
$$

where

$$
\mathbb{1}_{\{D_l(T_j)>0\}} = \begin{cases} 1 & : \text{ if } D_l(T_j)>0 \text{ (survival)} \\ 0 & : \text{ if } D_l(T_j)=0 \text{ (default)} \end{cases} \qquad l=1,\ldots,d
$$

We immediately infer the *annualized* default probability for each time to maturity T_j by

$$
\begin{aligned}
\tilde{\mathbf{q}} &= (\tilde{q}(T_1),\ldots,\tilde{q}(T_m)) \\
\tilde{q}(T_j) &= 1-\left(\frac{k(T_j)}{d}\right)^{\frac{1}{T_j}} \qquad j=1,\ldots,m
\end{aligned}
$$

5.2. Simulation Results

We provide evidence for the scaling law of time to maturity of the PLBM modeling results by the subsequent simulation outcomes. Since we look at a reasonably large number of firms (i.e. d = 7000 firms) and ensure convergence of simulation results by performing 30000 simulations we keep random errors at a minimum. We have chosen approximately optimal[‡] parameters of the loggamma distribution as listed in Table 1.

[‡]We only conducted an approximate nonlinear calibration of the parameters. This is a difficult procedure as the target function to be optimized is not analytic and relies on discrete events (simulated defaults). The simulations also require large amounts of computation time.

	d_0 (-)	a (-)	b (-)	α (-)	β (-)
discretization=1 yr	49.875	0.665	2.551	1.792	0.721
discretization=0.25 yrs	49.875	0.557	2.491	2.197	0.533

Table 1. Simulation parameters.

The parameter b in Table 1 exceeds a. Considering Eq. 17, this means that $X_l(T_j)$ can have a positive or negative sign. Thus upgrades of firms are possible as well as downgrades. A closer look shows that the resulting loggamma distribution of $X_l(T_j)$ is very asymmetric. The median is slightly negative (which means an upgrade), but the mean is positive due to the heavy upper tail (which means a downward move of the distance to default). In the long-term average, there is a slight tendency for a firm of being downgraded rather than upgraded.

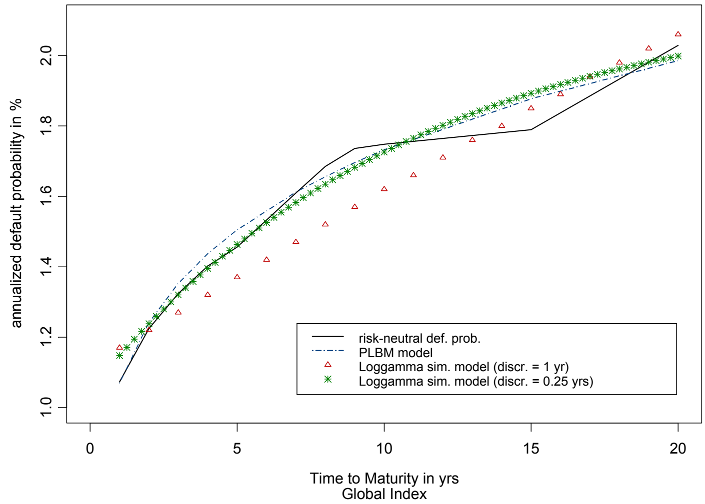

Figure 7. Model and simulation results of annualized default probabilities for the global index in dependence of the time to maturity. Annualized risk-neutral default probabilities are represented by the solid curve. Results of the PLBM model are represented by the dashed-dotted curve. Simulation results using loggamma distributed credit rating changes and discretizations of the time to maturity of one year and 0.25 years are shown by triangles and stars, respectively.

As indicated in Fig. 7 we observe that using loggamma distributed credit rating changes and a discretization of one year reveal deviations from modeling results for almost all times to maturity. These differences are likely to result from a discretization error since we compare discrete simulation outcomes with PLBM modeling results obtained from a continuous-time model. We can better describe the curvature and reduce deviations for most times to maturity by using a finer discretization. Moreover, we find from these simulation results that changes of the credit rating appear to follow a loggamma distribution as proposed in Eq. (17). On the whole, these simulation results provide an independent verification of the scaling law of time to maturity for annualized default probabilities. The heavy lower tail of the loggamma distribution is responsible for sudden defaults and seems to be an essential ingredient for a successful model.

6. CONCLUSION

Building on the access to industry yield data of different times to maturity and EDF values we develop a model that adjusts default probabilities to market consistent credit spreads based on a functional relationship and a scaling law of time to maturity. We model a firm's credit rating by using a continuous-time approach with a power-law scaling behavior with respect to the time to maturity. The empirical results of our study clearly

demonstrate that the proposed model enables to infer most of the credit spreads (market prices) from EDFs, verifying the statement made in the title of this paper. Independent of the time to maturity and of the sector index under consideration we find consistent results which are supported by values close to one for the model quality statistic G (i.e. $G \geq 0.85$).
We support the reliability and the efficiency of the model in an out-of-sample analysis and find that credit spreads (market prices) can be quite accurately predicted for short forecasting periods conditioned on the information available. We find that the model is independent of location and produces consistent results on the European bond market where data are scarce. We further observe that the model can be adequately used to approximate credit spreads on the corporate level but realize that in this application it reaches its limits which is reflected by relatively low values for the G statistic. Finally, we support and verify our heuristic model with the help of a Monte Carlo simulation study. We observe that credit rating changes are loggamma distributed and find promising evidence that annualized default probabilities indeed follow a scaling law with respect to the time to maturity.

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