Re-sequencing optics instruction to first build conceptual understanding in introductory courses

Sarah L. McGregor[a] and Jacob Pleasants[b]

[a]Keene State College, Keene, NH, USA
[b]University of Oklahoma, Norman, OK, USA

ABSTRACT

It is common instructional practice to introduce foundational concepts such as refraction and lenses in optics instruction beginning with a diagram and an equation, often followed by demonstrations, problem sets, and experiments. This common instructional approach is consistent with how experts understand these phenomena, with mathematical relationships deeply integrated with conceptual understanding and thus the formulas are indecipherable from the core concepts. However, many students do not see the interwoven nature of the mathematics and concepts, and instead see our pivotal mathematical relationships, such as Snell's Law or the Lensmaker's Equation, as black boxes that provide an answer or a formula to be memorized, but not understood. These issues are only enhanced for students exhibiting math-anxiety. In this paper, we present an approach for presenting optics concepts in a way that promotes a student's marriage of conceptual and mathematical knowledge by re-sequencing often used instructional activities. By placing conceptual understanding in the foreground, we can provide students with a rich set of experiences around the phenomena first and then layer formal mathematics ideas on afterwards. In this way, the mathematics become a validation tool for student's conceptual knowledge. We provide general guidelines for the adoption of this instructional approach, followed by more detailed examples of how this instructional method could be implemented for two foundational optics phenomena: Snell's Law and the Lensmaker's Equation.

1. INTRODUCTION

Optics instruction often begins with foundational concepts such as refraction and the behavior of lenses. These concepts form the foundation upon which more complex ideas are built,[1] and are therefore typically appear early in instruction. Like most fundamental physics ideas, these concepts include mathematical representations and relationships (e.g., Snell's Law and the Thin Lens Equation). As such, those equations are often presented to students early during instruction and are typically accompanied by sets of practice problems.

This common instructional approach is illustrated by how refraction is often first presented to students. Students are typically presented with a ray diagram (see Figure 1) that shows the difference in the the angle of incidence (θ_1) and refraction (θ_2) for a refracted ray of light. Snell's Law is then introduced, expressed as $n_1 \sin\theta_1 = n_2 \sin\theta_2$, along with a definition of the index of refraction (n) as the ratio of the speed of light in a vacuum to the speed of light through that medium: $n = \frac{c}{v}$. Students might also be provided a justification for refraction based on the change in speeds across the boundaries, and they might also see some demonstrations of the principle in action. In short order, students are given sets of mathematical exercises and perhaps a laboratory investigation where students experimentally verify Snell's Law by measuring incident and refracted angles.

This common instructional approach is consistent with how experts understand refraction, in that mathematical representations are seamlessly interwoven with the core concepts. Unfortunately, for many students, mathematical relationships such as Snell's Law rarely carry the same conceptual meaning as they do for experts. For the expert, mathematical relationships are deeply integrated with conceptual ideas,[2–4] and are used as cognitive tools to make sense of a variety of situations and applications.[5,6] In contrast, for many students the fundamental equations are simply "formulae" to be memorized and recalled. Although they might learn how

Further author information: (Send correspondence to S.L.M.)
S.L.M.: E-mail: sarah.mcgregor@keene.edu
J.P.: E-mail: jacob.pleasants@ou.edu

Optics Education and Outreach VII, edited by G. Groot Gregory, Anne-Sophie Poulin-Girard, Proc. of SPIE Vol. 12213, 122130A
 · doi: 10.1117/12.2632583

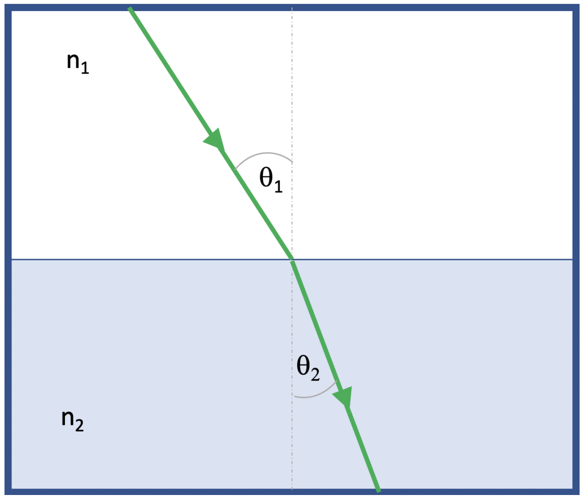

Figure 1. Typical diagram used to introduce Snell's Law to students.

to carry out the set of calculations that they encounter in problem sets, their knowledge of the mathematics is often divorced from conceptual principles.[3,4,7–11] Students' abilities to carry out calculations can therefore mask conceptual misconceptions that they might hold.[5,12] These issues have been documented not only for science in general but optics concepts in particular.[10,13,14]

Another example that illustrates these problems is the Lens Maker's Equation. Here students are usually provided the equation $\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$, along with a diagram (See Figure 2). In this case the mathematical relationship is even less intuitive (why are the focal length and radii in the denominator?) and can easily become a "magical" formula to be used but not understood. Students are often confused about the meaning of the radius of curvature, don't understand why there are two surfaces, are unsure why the radius comes into play at all, don't have a conceptual grasp of the role of the index of refraction of the lens, and thus see the equation as a purely black box to insert numbers into and get an answer so they can move on. For those students with math-anxiety, this can become panic inducing. Showing students a mathematical derivation of the equation is highly unlikely to solve the problem; a derivation presupposes a conceptual understanding that students have not yet obtained. Along with confusion around the purpose and use of the equation, the problem is often compounded for students as efforts to look it up online provide variations of the equation (e.g. $\frac{1}{f} = (n-1)\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$, $\frac{1}{f} = \frac{n-n_o}{n_o}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$).

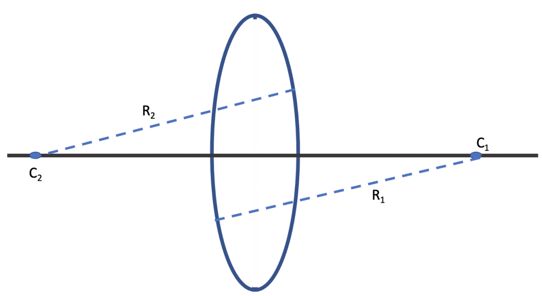

Figure 2. Typical diagram that is presented to students when talking about the Lensmakers Equation.

The ways that students understand (or fail to grasp) the mathematical aspects of physics concepts is a concern for many instructors. On the one hand, mathematics is an integral part of physics and cannot be avoided. On the other hand, an over-emphasis on calculation is potentially unproductive in terms of promoting conceptual understanding. An emphasis on mathematics can also cause students to exit STEM majors, especially for those who have anxieties or low-self efficacy beliefs towards mathematics.[15–17] Because mathematical ideas cannot (and should not) be avoided, the goal is therefore to determine how to present the mathematics behind key physics concepts in a way that avoids the typical pitfalls. The desired state is one in which students are not merely memorizing algorithms for solving narrow types of physics problems, but rather one in which students are developing more "expert-like" knowledge in which conceptual understanding is integrated with the mathematics.[3,4,8,9,11] In other words, the goal is to make the mathematics **meaningful** for students rather than just another piece of information that they need to assimilate.

In this paper, we present an approach that we have developed for presenting optics concepts in a way that promotes students' integration of conceptual and mathematical knowledge. The conceptual framework underlying this approach is summarized in Figure 3. As shown in that framework, our objective is that students develop an understanding of science concepts alongside their mathematical representations in ways that allow them to apply both to relevant phenomena. That kind of integrated understanding is reflective of expert-like knowledge. Traditional instructional approaches tend to move rapidly from phenomena to mathematical descriptions. In contrast, our instructional approach develops conceptual understanding prior to layering on mathematical representation and analysis.

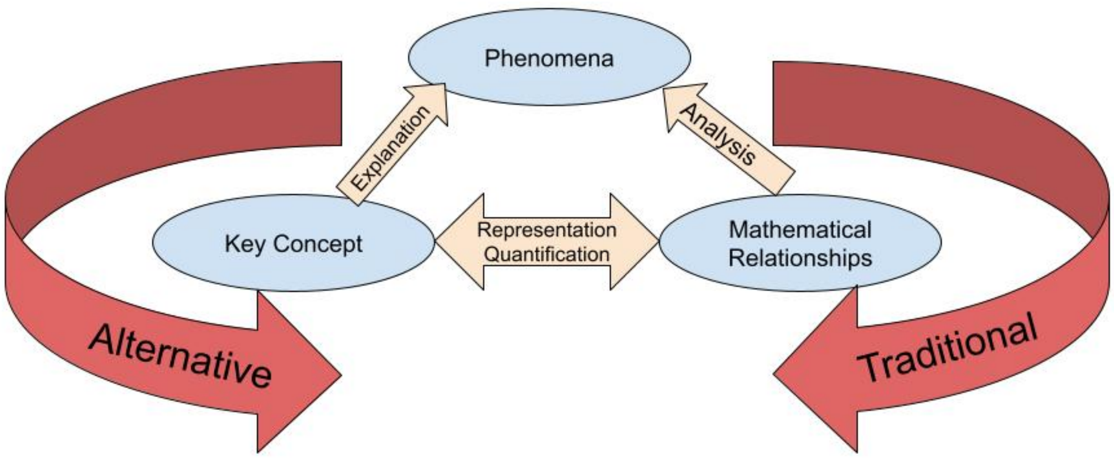

Figure 3. General form of the instructional model. The goal of instruction is for students to have a well-integrated understanding of the science concepts, the mathematical representations of those concepts, and how those ideas make sense of relevant phenomena. Our instructional model begins with the phenomena and then proceeds to establish the concepts, followed by the mathematics. A more traditional approach tends to move in the opposite direction.

In the following sections, we first provide general guidelines for implementing this process and then illustrate how we have put this model into practice for two foundational mathematical relationships in optics: Snell's Law and the Lens Maker's Equation. In our approach, instead of leading with mathematics, we instead provide students a set of rich experiences with phenomena. We then introduce key canonical physics concepts in qualitative terms in strategic ways so that students use those ideas to make sense of the situations at hand.[12] We then layer the formal mathematics ideas on topic of that conceptual foundation when the need for quantitative precision arises. In this way, the instructional sequence supports the use of mathematics as a conceptual resource[6] rather than as a set of formulae-to-be-memorized.

2. GENERAL GUIDELINES FOR ADOPTION

In this section we identify key instructional decisions as well as helpful principles towards implementing this concept-first framework. These guidelines and principles have assisted us in our own design processes and provide a broad sense of how the above instructional model could be implemented.

2.1 How Should I Select the Initial Phenomena?

Our instructional model begins by having students explore phenomena, and thoughtful selection of those phenomena is essential. Of greatest importance: the science concepts and mathematical relationships that students learn during instruction need to help students **make sense** of those initial phenomena. Students' initial engagement with the phenomena, therefore, should raise questions and curiosities that the science concepts will help them address. General guidelines include:

1. Select introductory phenomena that are not fully explicable without the use of the concepts that will be subsequently developed.
2. Let students explore those phenomena and raise questions; there is no need to provide a full explanation yet - the goal is to create a need to learn new things.
3. If needed, draw students' attention to key aspects of the phenomena that the science concepts are going to elucidate.
4. Ensure that the concepts introduced later in the sequence will enable students to provide a satisfactory answer to the questions that arise during their exploration.

2.2 How Should I Introduce the Science Concepts?

Before introducing mathematical relationships, our instructional model addresses the conceptual ideas underpinning those relationships. When introducing these new ideas, it is essential that students can see how the ideas **make sense** of the phenomena they previously explored. Extraneous details, while perhaps interesting, are unlikely to be remembered by students unless they are relevant to phenomena that the students have encountered. Some guidelines are as follows:

1. Tempting as it might be to tell students everything that one might want to know about the topic at hand, instead limit the new information to the ideas that are most relevant to the phenomena students have encountered. Other ideas can come later!
2. Ask students to apply newly-introduced ideas to the phenomena they explored and the questions that were raised during that explanation.
3. If possible, introduce related phenomena that the new ideas are also helpful for understanding

2.3 How Should I Introduce the Mathematical Relationships?

The mathematical relationships are important, and the goal of our approach is to help students recognize them as such! We want students to see how the mathematical ideas provide useful thinking tools that allow them to analyze phenomena of interest and generate new insights. The last thing we want to do is assign a series of exercises that students regard as mere "tasks to grind through." Some guidelines on making the mathematics meaningful:

1. Identify questions about the phenomena at hand that are quantitative in nature. This establishes a need for mathematical analysis.
2. If possible, have students collect quantitative data about the phenomena that will lead them to develop the targeted mathematical relationship.
3. If the relationship is not easily developed from data, help students see how the formal equation represents conceptual relationships between quantities that students have already encountered. This approach is illustrated in the Lensmaker's Equation example described below. Before providing the full equation, helping students recognize relationships (e.g., inverse) between the key quantities will help students recognize even complex equations as meaningful representations of concepts rather than magical formulae.
4. Utilize the mathematics to analyze the phenomena students have encountered and ensure that those analyses lead to valuable insights. The calculations should serve some kind of purpose, and not simply be calculations for their own sake

3. EXAMPLE IMPLEMENTION OF APPROACH FOR MATHEMATICAL CONCEPTS

To assist instructors who are interested in taking up, applying, and extending our instructional model, we now provide a more detailed explanation and examples of how the model shown in Figure 3 and the guidelines discussed above could be implemented in an optics classroom. Prior to starting this sequence, instructors need to first identify the concepts they will target and carefully consider the introduction of the mathematical relationships as outlined in the previous section (Section 2). These careful, thoughtful decisions become the guiding principles for the students' exploration into understanding the phenomena.

Once these instructional decisions have been made, the model outlined below can be used as one possible instructional strategy for implementation. This model should not be considered rigid, but fluid and customizable for each situation and topic, with broad definitions of the proposed phases, expansion or addition of activities, or repetition of phases as needed. Consider the below phases as example guideposts that can and should be modified for each situation, but that also provide a starting point and initial pathway for instructional sequencing. Most important is to provide a safe environment for students to explore, observe, predict, quantify and ultimately verify optics phenomena while providing only the information that is needed to guide the students to the next step. Only once their conceptualization and internalization of the concept is complete are the mathematics introduced.

Possible Implementation Strategy

Phase 1: Students explore the phenomena

Phase 2: Instructor provides restrained guidance

Phase 3: Students explore, predict, observe and quantify

Phase 4: Determination of mathematical relationship or expressions

Phase 5: Students verify the mathematics

Students are first introduced to the phenomena in question through a guided exploration or experiment. This can be purely observational and qualitative or could also be quantifiable. No new lecture material is provided at this juncture, only instructions for the exploration. Only after students have explored and the observations/results have been discussed is more formal guidance provided for the class (phase 2). The key here is to provide enough information that students understand what they have observed, but not the entire picture yet. This could occur via traditional lecture, discussion, board work, or guided questions, but in any case students should feel empowered that they understand more (but not all) of the phenomena now.

In the third phase students take the phenomena one step further. If seen from an experts point, this could be likened to a parameter study for an equation. How do the results change in different setups/scenarios? Students can again explore phenomena or, at this point in the sequence, they could also begin to predict outcomes from experiments. Students are guided through experimentation to measure and test against their predictions and outcomes. Only at the end of Phase 4 are students presented with the formal mathematics related to the phenomena under investigation. Either they have empirically derived the equation, or perhaps derived component relationships for the quantities in question (and then are provided the final equation by the instructor). The key point is that students have the equation in hand only AFTER they understand how the different variables affect the quantity in question. The final step has students practicing the mathematics and verifying their equation. Now that they understand conceptually, they use that conceptual understanding as the rule, and 'test' the mathematics to see if it follows their rules.

The two examples described in the next section illustrate how our instructional model above could be implemented for two well known mathematical optics concepts that are commonly taught in introductory optics courses: Snell's Law and the Lens Maker's Equation. Although the instructional sequences are not identical, we identify how they align within the guiding phases outlined above as an example of how the instructional model provides a flexible framework. The specific examples that we describe in this paper were designed for a general education introductory optics class (2-hr time block) at a liberal arts school. However, the overarching teaching model is easily adaptable to other educational settings as well as different science content.

3.1 Example of the Instructional Framework 1: Snell's law

The first example is described in greater detail in McGregor 2022, but below is a brief summary that illustrates how each of the above steps is realized in the classroom. We've streamlined the instructional sequence here to make the decisions clearer, but reiterate the the entire instructional sequence contains several offshoots, not described here, that weave around the phases outlined above. The first step for the instructor, pre-class, is to identify the conceptual ideas that students may struggle with, that are the most important for understanding the phenomena, and that form a natural progression to the mathematics. For Snell's law we summed these conceptual points up into 3 key points, found below.

- light refracts at boundaries of materials with different indexes of refraction
- the side with the larger index of refraction has the smaller angle
- the larger the difference in indexes of refraction, the larger the difference in the angles

3.1.1 Phase 1: Students explore the phenomena

As a sequence within an optics class, students would have already been exposed to the idea of light moving in a straight line unless it interacts with an object, and ideas such as the law of reflection would have already been discussed. However, this sequence could be modified for a 'one-off' class on Snell's law if needed.

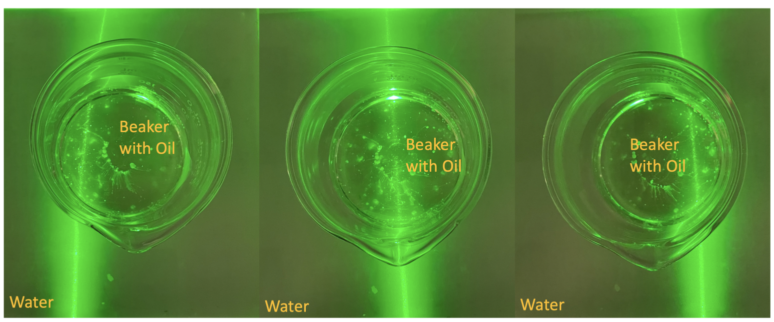

Figure 4. Example of student experiments shining a laser through a beaker filled with vegetable oil (n= 1.47) placed in a container of water (n=1.33)

Students are first given an experimental setup where they are to shine light through a square tub containing colored water that is labeled a 'number'. In reality these numbers are the indexes of refraction, but students are not told this. Students are first told to shine the laser pointer through the large square plastic tub to confirm that they can see a straight beam of light. A small beaker is then placed inside the larger tub. This beaker holds another material (oil for example) marked with another 'number'. Students observe the light as it passes through different beakers with different numbers, and make observations about the numbers and the path of the light. In particular we try to ensure that students come up with observations about the following:

⇒ Where does the light 'bend'?

⇒ Does the light bend the same in all scenarios?

⇒ When does the light not bend ?

3.1.2 Phase 2: Instructor provides restrained guidance

In this next phase the instructor introduces useful science ideas (but not a "comprehensive" scientific account of refraction!) that explain what is happening. First the facts of the observations above reiterated. Then we discuss why the 'bending' of the light beam happens at the boundary only (and not in all situations) and define the 'bending' as refraction. We finally define the 'number' as the index of refraction (n=c/v) and that it is a measure of the speed of the light. Using diagrams such Figure 1 to discuss that as the light enters a medium and the speed changes, the wavelength also has to change, and thus the angle changes in response. Although it is tempting to proceed further here, it is important to stop at this point to allow students to work through the next set of questions themselves.

3.1.3 Phase 3: Students explore, predict, observe and quantify

Armed with their new knowledge students are now presented with a set of half moon dishes filled with liquids of different indexes of refraction. Placed flat-side together on a polar-grid students are instructed to shine the laser through the two half moon dishes, recording the incident and refracted angles (see Figure 5). While exploring, students should be guided to answer the questions below.

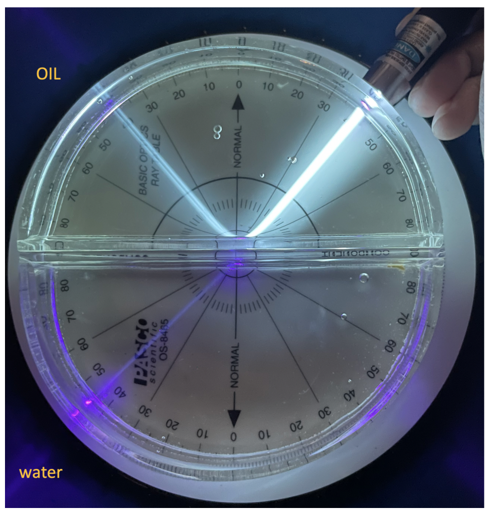

Figure 5. Example of student experiment shining a laser through two halfmoon dishes containing materials with different indexes of refraction: vegetable oil (n=1.47) on the top and water (n=1.33) on the bottom.

- ⇒ Is the angle of incidence θ_i the same as the angle of refraction θ_r ?
- ⇒ In terms of what you see when you look at the refraction, does it matter which direction the light is going (in other words, which side the light is incident on)?
- ⇒ When you compare the angle of incidence to the angle of refraction, what patterns do you see in terms of which one is larger and which is smaller?

3.1.4 Phase 4: Determination of mathematical relationship or expressions

Students will now have created a list of observations about the incident and refracted rays as they entered and exited the half moon dishes. Now work with the students to determine what type of mathematical relationship might exist between the incident and reflect ray angle. You can guide students to look beyond just simple arithmetic to consider trigonometric functions by looking at the wave-crest diagrams (Figure **??**) once more. Student can empirically determine (via trial and error) which 'function' works for comparing the incident and refracted angles.

3.1.5 Phase 5: Students verify the mathematics

Once students have arrived at Snell's law, provide them with examples to mathematically try. Here students can use the equation to make predictions, and then test the calculated angles using the setups in front of them. Students are now testing the math, and proving it with their observations and experimentation, instead of the other way around. They have ownership of the math, and an internalization of how the incident and refracted angles relate to each other.

3.2 Example Instructional Framework 2: Lens Maker's Equation

The second example for the implementation of the instructional framework concerns the Lens Maker's Equation. Again, the first step for the instructor, pre-class, is to determine the key concepts to target. For The Lens Maker's Equation those concepts can be summarized as:

- The Shape of a lens is what causes the light to focus
- Radius of Curvature has a direct relationship with the focal length
- The index of refraction has an inverse relationship with the focal length

In this case we will be focusing on plano-convex lens in order to simplify the Lens Maker's Equation. This will allow students to get used to concepts of lenses, and the variables that affect lenses, without getting muddled in 2 curved surfaces. Additional lessons can be devoted to understanding the effects of adding a second curved surface, or investigating what happens in situations with diverging lenses.

3.2.1 Phase 1: Students explore the phenomena

Students are supplied with a set of different shaped prisms, including rectangular prisms and semi-circular prisms. They are also provided with a ray-box to emit parallel light beams. Students are given the instructions to first place the rectangular prism in the path of the light beams, and explain, draw what happens (see Figure 6A. Make sure that student's pay attention to the following two ideas.

⇒ Does Snell's Law still apply?

⇒ Where is the light bending? (The same as before?) (important for understanding lens diagrams later)

Next, students are to explore light as it goes through a semi-circular prism. Make sure they set up the prism so that the parallel light source is entering the flat side and exiting the curved side. Allow students to play with the prism and write down any observations. Suggest to students to lean into their understanding of Snell's law to draw/explain why the parallel light beams behave differently with the semicircular prisms than they do with the rectangular ones.

3.2.2 Phase 2: Instructor provides restrained guidance

In this example, Phase 2 is primarily about discussing the solutions and drawings that students have created to explain why the light converges for the semicircular prism, focusing students on the changing direction of the Normal to the surface. In addition, make sure to build from concepts from previous lessons, like those from Snell's law concerning which side of the boundary will have a larger angle (the side with the lower index of refraction), which in this case is the air. In addition, make sure to define the focal point and focal length of our semicircular prisms, as well as introducing the idea of the radius of curvature.

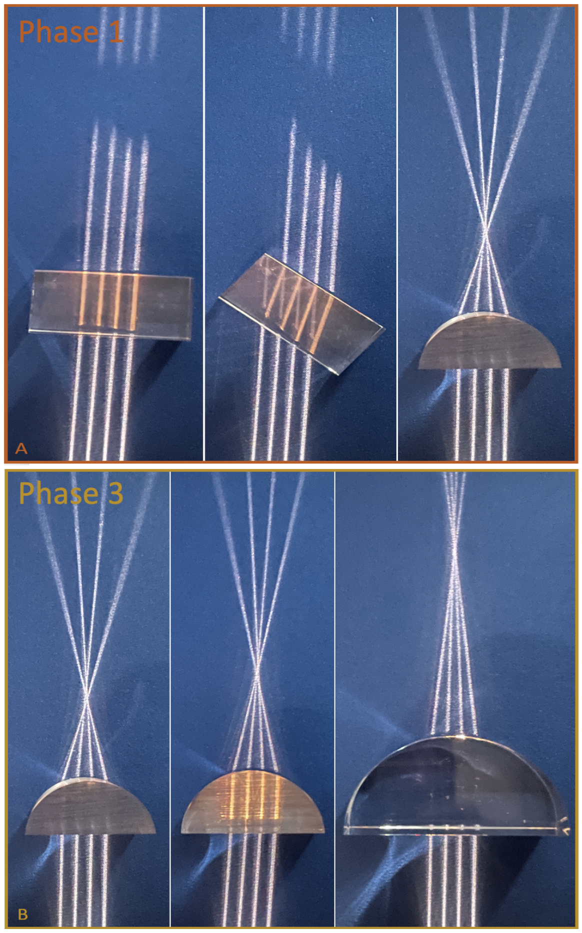

Figure 6. Example of what students will see when exploring how the shape of materials affects the light that is transmitted. The top panel (A) shows the materials used for Phase 1, when students are solidifying the idea that if the lens is curved light will focus to a point. The bottom panel (B) shows what students could see during Phase 3 when using acrylic prisms of different sizes, showing how the Radius of curvature is directly related to the focal length.

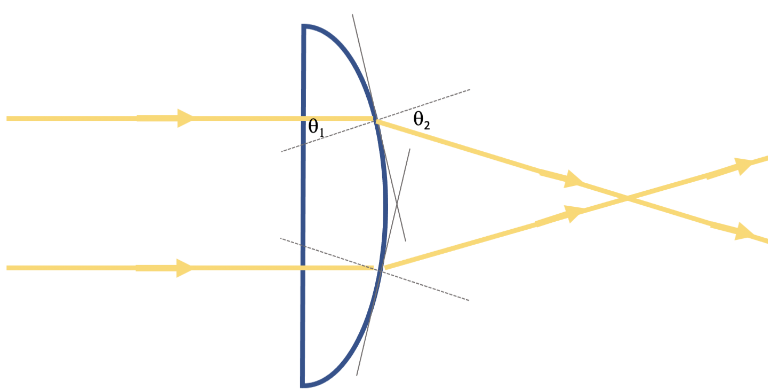

Figure 7. Example drawing created in conjunction with students about why parallel light beams (yellow) converge to the focal point when using the semi-circular prism.

3.2.3 Phase 3: Students explore, predict, observe and quantify

Now that the instructor has gone over why the light focuses for a semi-circular prism, we can have the students take this one step further. Here, instead of jumping straight to exploration, its helpful to have students predict the outcome based on their knowledge of refraction and the information you have just presented, and then have students check to see if their predictions hold true. Students will look at 2 different situations, with the goal that they will be able to answer the following questions.

⇒ As the prism becomes more curved, what happens to the focal length?

⇒ As the material of the prism changes, what happens to the focal length?

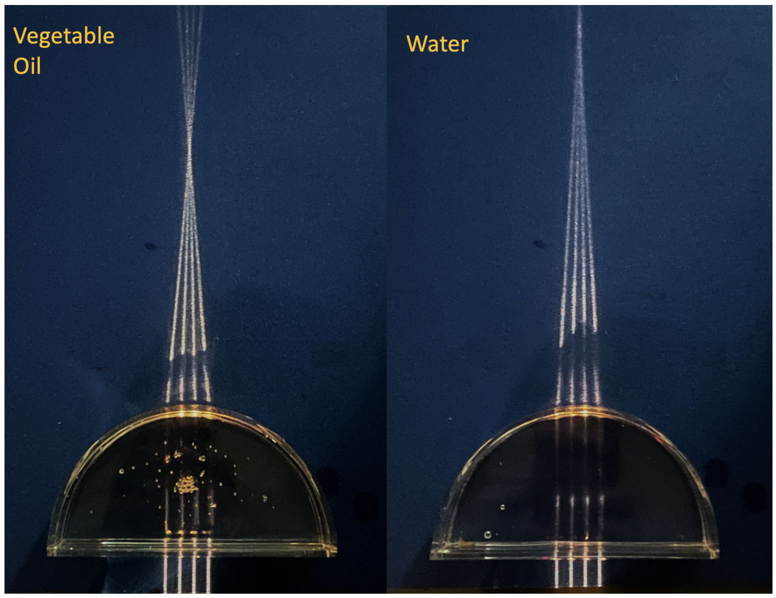

Figure 8. Example of student experiment using 2 halfmoon dishes that are the same size but filled with water and oil, showing the inverse relationship between the index of refraction and the focal length)

Present the students with several semi-circular prisms of varying sizes made of the same material and then ask them if they believe, based on the discussion, if each prism will have the same focal length. Most students agree that they will be different, and when they do, have students predict which of the prisms will have the longest and shortest focal length. Students can work in groups to create explanations and drawings to back up their choices. Only once the group has developed their explanation can students verify their prediction. Let them test each of the semi-circular prisms, modifying their explanation as needed until they determine that the changing normal direction due to the radius of curvature means that the more curved the surface (smaller radius of curvature) the smaller the focal length (see Figure 6B).

Repeat this process for a second situation where students are provided prisms of the same size but different indexes of refraction, (or you can use the half moon dishes from the Snell's Law lesson with oil and water again as seen in Figure 8). Make sure that the radius of curvature of whatever you use are the same. Have students work through their prediction and diagrams before testing once more. As they test, allow them to modify their explanation as needed until they determine that the increase difference in the indexes of refraction of the material and air (therefore the higher the index of refraction of the material) means a larger refracted angle on the surface and therefore means a smaller focal length.

3.2.4 Phase 4: Determination of mathematical relationship or expressions

At this point students have identified some general relationships between the radius of curvature and the index of refraction of the material with the focal length. Here we turn that into an equation. Work with the students to turn their verbal explanations into a mathematical relationships: inversely proportional and proportional changes. Only once they are solid with the relationships should the instructor introduce the lens makers equation in canonical form. Once introduced, the instructor can finally define the lens makers equation. Noting that the R1 term vanishes in the case presented in class (because R1 for a planar lens is infinite), reiterate how the equation reiterates what the students already know. It's tempting to expand further, but it is important to not overload the students here.

Note: for students with math anxiety it is important to go slow here with the fractions. Fractions can cause significant confusion for the math phobic, so go slowly through several examples on the mechanics of solving the math here if you have any math-phobic or math-anxiety students.

3.2.5 Phase 5: Students verify the mathematics

Despite the thickness of the semi-circular prisms, because they are essentially plano-convex lenses the first interface is negated. We can use the Lens Maker's equation to verify. Have students measure the radius of their prisms and knowing the index of refraction they can verify the focal lengths, showing that the smaller radius prisms (more curved) have smaller focal lengths. Once again, this focus on the mathematics after conceptualization allows the formula and equations to become affirmations of students' scientific understanding of the phenomena. Only after students have run through several mathematical examples and trust that the equation validates their conceptual understanding can the instructor bring in more information. For example, a discussion would easily follow of what would happen if the first interface was curved instead of flat.

4. DISCUSSION

As outlined previously, the most important aspect of this instructional approach is to allow students to build a conceptual understanding of the phenomena before the introduction of the mathematics. Allowing students to build a foundational understanding and internalize the results before they plug things into a calculator provides them ownership of the mathematics and makes math meaningful in ways that the equations by themselves rarely provide. In this way, the mathematics enhances and verifies their own understanding. It is not expected that all concepts will be able to be addressed using the example instructional sequence provided in Section 3, but it is likely most can be taught from a modified version stemming from a focus on conceptualization before mathematics. We believe that the overall model shown in Figure 3, guiding questions discussed in Section 2 and modified version of the implementation strategy outlined above offer a useful set of guidelines and principles when developing an instructional plan that places conceptual knowledge first and makes mathematical ideas meaningful.

ACKNOWLEDGMENTS

REFERENCES

[1] Hestenes, D., Modeling theory for math and science education. In R. A. Lesh, P. L. Galbraith, C. R. Haines, & Hurford (Eds.), *Modeling students' mathematical modeling competencies* (pp. 13–41),(2010), https://doi.org/10.1007/978-1-4419-0561-1_3.

[2] Pospiech, G., Framework of mathematization in physics from a teaching perspective. In G. Pospiech, M. Michelini., & B. Eylon (Eds.), *Mathematics in physics Education* (pp. 1-33). Springer. (2019), https://doi.org/10.1007/978-3-030-04627-9_1.

[3] Sherin, B. L., How students understand physics equations. *Cognition Instruct*, **19** (4), 479-541.,(2001), https://doi.org/10.1207/S1532690XCI1904_3.

[4] Zhao, F., & Schuchardt, A., Development of the Sci-math Sensemaking Framework: categorizing sensemaking of mathematical equations in science. *Int J STEM Educ*, **8** (1), 1-18., (2021), https://doi.org/10.1186/s40594-020-00264-x.

[5] Bransford, J., Bransford, J. D., Brown, A. L., & Cocking, R. R. "How people learn: Brain, mind, experience, and school." *National Academies Press.*, (1999).

[6] Hammer, D. Student resources for learning introductory physics. *Am J Phys*, **68** (S1), S52-S59. (2000), https://doi.org/10.1119/1.19520.

[7] Bing, T. J., & Redish, E. F., ”Analyzing problem solving using math in physics: Epistemological framing via warrants.” *Physical Review Special Topics-Physics Education Research*, **5**(2), (2009), https://doi.org/10.1103/PhysRevSTPER.5.020108.

[8] DiSessa, A. A. Toward an epistemology of physics. *Cognition Instruct*, **10**(2-3), 105-225,(1993), https://doi.org/10.1080/07370008.1985.9649008.

[9] Eichenlaub, M., & Redish, E. F. ”Blending physical knowledge with mathematical form in physics problem solving.” In G. Pospiech, M. Michelini, B.-S. Eylon (Eds.), *Mathematics in Physics Education* (pp. 127–151). Springer. (2019), https://doi.org/10.1007/978-3-030-04627-9_1.

[10] Hammer, D., Epistemological beliefs in introductory physics. *Cognition Instruct*, **12** (2), 151-183 (1994), https://doi.org/10.1207/s1532690xci1202_4

[11] Uhden, O., Karam, R., Pietrocola, M., & Pospiech, G., Modelling mathematical reasoning in physics education. *Science Education*, **21** (4), 485-506., (2012), https://doi.org/10.1007/s11191-011-9396-6.

[12] Edelson, D. C., Learning-for-use: A framework for the design of technology-supported inquiry activities. *J Res Sci Teach*, **38**(3), 355-385,(2001), https://doi.org/10.1002/1098-2736(200103)38:3<355::AID-TEA1010>3.0.CO;2-M.

[13] John, M., Molepo, J. M., & Chirwa, M., Secondary school learners’ contextualized knowledge about reflection and refraction: a case study from South Africa. *Res Sci Technol Educ*, **36** (2), 131-146., (2018), https://doi.org/10.1080/02635143.2017.1395331.

[14] Elby, A. Another reason that physics students learn by rote. *Am J Phys*, **67**(S1), S52-S57. (1999), https://doi.org/10.1119/1.19081.

[15] Ellis, J., Fosdick, B. K., & Rasmussen, C., ”Women 1.5 times more likely to leave STEM pipeline after calculus compared to men: Lack of mathematical confidence a potential culprit.” *PloS one*, **11** (7), 1-14. (2016), https://doi.org/10.1371/journal.pone.0157447.

[16] Graham, M. J., Frederick, J., Byars-Winston, A., Hunter, A. B., & Handelsman, J., ”Increasing persistence of college students in STEM.” *Science*, **341**(6153), 1455-1456. (2013).

[17] Xie, Y., Fang, M., & Shauman, K., STEM education. *Annu Rev Social*, 41, 331-357.,(2015) https://doi.org/10.1146/annurev-soc-071312-145659.

[18] ,McGregor,S.L. & Pleasants,J., ”Shedding light on boundaries: re-sequencing Snell’s law instruction to first build conceptual understanding”, *Physics Education*, **57**(5),055018. (2022). https://doi.org/10.1088/1361-6552/ac6eb4