Keys to structuring a Fourier optics course

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ABSTRACT

The teaching of Fourier optics is greatly facilitated by the introduction of certain key tools—conceptual and analytical—and by a suitable sequencing of topics. Points addressed in this paper include the following: (a) Students of electrical engineering easily relate to an analysis of wave field propagation based on the angular spectrum concept, which can be exact but also leads in simple approximations to Fresnel regime expressions. (b) The notion of inverse propagation, where the wave field is in effect propagated backwards in time (possibly to a virtual or effective source distribution), can be an extremely useful tool in certain optical system analysis problems. (c) The preferable canonical systems for examining the Fourier transform property of a lens have the lens imaging a point source and the object in either the converging or diverging beam; the FT appears in the plane conjugate to the source point. (d) Coherent imaging systems are often analyzed incorrectly in Fourier optics courses, and adequate attention is rarely given to systems of a general nature. These deficiencies can be overcome by proper sequencing of systems analyzed and by including the analysis of a general system characterized by entrance and exit pupils.

Keywords: Fourier optics, optics education, coherent imaging, optical Fourier transform, inverse propagation

1. INTRODUCTION

The teaching of Fourier optics, dating back more than 30 years now, is identified with the application of linear systems theory to optics.[1] Topics covered typically include optical wave field representation, wavefront propagation, the Fourier transform property of lenses, coherent and incoherent imaging, holography, and optical information processing.

In the teaching of Fourier optics, as in the introductory teaching of any subject, the subject matter should be covered adequately with a minimum of complexity. This objective translates into the following general rules for structuring a course: (1) identify the important aspects of the subject and cover them; (2) determine the optimum order in which they should be covered; and (c) do not make the material more difficult than necessary.

For 25-plus years now I have taught a course on Fourier optics and sought to achieve the basic objective stated above. Recently, in working to complete a textbook on the subject,[2] I have, I think, found a particular set of keys that is especially important to that achievement. It is the purpose of this paper to share those keys. In terms of content, the course is relatively standard: it covers optical wave field representation, wavefront propagation, lenses (imaging and Fourier transform property), coherent imaging, incoherent imaging, and a subset of topics that can include an introduction to coherence theory, holography, and optical signal processing. I have used two standard texts extensively in the course, Goodman's *Introduction to Fourier Optics*, and Gaskill's *Linear Systems, Fourier Transforms, and Optics*.[3,4] In the following sections I discuss some of my principal classroom deviations from those texts.

2. ANGULAR SPECTRUM EMPHASIS

The traditional approach to diffraction theory is via Huygens' principle and the Fresnel-Kirchhoff diffraction integral formulation. I find it much faster, and also more appealing to students of electrical engineering (who generally have considerable background working with the Fourier integral transform), to approach wave field propagation via the angular spectrum of plane waves. It is first necessary to introduce the basic complex amplitude model for a plane wave:

$$\mathbf{U}_{pw}(x,y,z) = \exp[jk(\alpha x + \beta y + \gamma z)] , \quad (1)$$

where α, β, γ are the direction cosines associated with the wave. It is easy to show that free-space wavefront propagation involves a linear and shift-invariant operation. Thus, with reference to Fig. 1, one can write

$$\mathbf{U}(x,y;z) = \mathbf{U}(x,y;0)**\mathbf{h}_z(x,y), \tag{2}$$

where z denotes the plane-to-plane propagation distance and where ** denotes two-dimensional convolution. Equivalently, in the spatial frequency domain,

$$\hat{\mathbf{U}}(u,v;z) = \hat{\mathbf{U}}(u,v;0)\mathbf{H}_z(u,v), \tag{3}$$

where $\hat{\mathbf{U}}$ and $\mathbf{H}_Z$ denote the Fourier transforms of U and $\mathbf{h}_Z$, respectively. The trick is to find the propagation kernel $\mathbf{h}_Z(x,y)$ or, alternatively, its Fourier transform $\mathbf{H}_Z(u,v)$. Through the formalism of the Angular spectrum, the later approach is much easier. Following Goodman, U(x,y;0) is first written as an inverse Fourier transform:

$$\mathbf{U}(x,y;0) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{\mathbf{U}}(u,v;0)\exp\left[j2\pi(ux+vy)\right]du\,dv \, . \tag{4}$$

The similarity between the 2-D Fourier kernel and the representation of a plane wave in the z=0 plane is then exploited to find the superposition of plane waves after plane-to-plane propagation through a distance z:

$$\mathbf{U}(x,y;z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left\{\hat{\mathbf{U}}(u,v;0)\exp\left[j2\pi\frac{z}{\lambda}\sqrt{1-(\lambda u)^2-(\lambda v)^2}\right]\right\}\exp\left[j2\pi(ux+vy)\right]dudv \, . \tag{5}$$

The transfer function $\mathbf{H}_Z(u,v)$ is thus identified as

$$\mathbf{H}_z(u,v) = \exp\left[j2\pi\frac{z}{\lambda}\sqrt{1-(\lambda u)^2-(\lambda v)^2}\right]. \tag{6}$$

At this stage, I depart from Goodman's analysis. Binomial approximation of the square root immediately yields the Fresnel-regime transfer function,

$$\tilde{\mathbf{H}}_z(u,v) = \exp\left(j2\pi\frac{z}{\lambda}\right)\exp\left[-j\pi\lambda z\left(u^2+v^2\right)\right], \tag{7}$$

which is easily inverse transformed to yield the corresponding approximate propagation kernel:

$$\tilde{\mathbf{h}}_z(x,y) = \frac{\exp(jkz)}{j\lambda z}\exp\left[j\frac{k}{2z}\left(x^2+y^2\right)\right]. \tag{8}$$

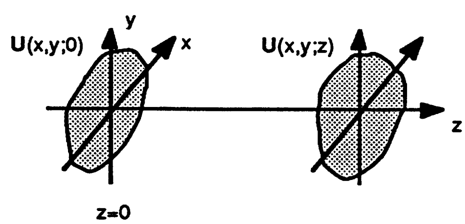

Fig. 1. Plane-to-plane propagation of a complex wave amplitude distribution.

This latter result is identical with that obtained via the traditional Fresnel-Kirchhoff diffraction integral formulation, but without the need to introduce Green's theorem and Green's functions.[5] An exact result is also obtainable by direct inverse Fourier transformation of Eq. (6).[6]

3. INVERSE PROPAGATION

The relationship between U(x,y;0) and U(x,y;z) can be calculated in either direction (for monochromatic waves, there is, in fact, no sense of time or, therefore, actual direction of propagation). To calculate U(x,y;z) given U(x,y;0), Eq. (2) or Eq. (3) can be used. To calculate U(x,y;0) given U(x,y;z), on the other hand, either of the following equations can be used:

$$\mathbf{U}(x,y;0) = \mathbf{U}(x,y;z) ** \mathbf{h}_z^*(x,y) \tag{9}$$

or

$$\hat{\mathbf{U}}(u,v;0) = \hat{\mathbf{U}}(u,v;z)\mathbf{H}_z^*(u,v), \tag{10}$$

the single asterisk denoting complex conjugation. As will be seen later, inverse propagation can be used to great advantage in the analysis of certain optical systems.

4. LENSES

As does Gaskill, I find it best simply to argue that a thin spherical lens is designed to map an expanding spherical wave into a converging spherical wave in accord with Fig. 2 and is thus represented, in the Fresnel regime, by complex amplitude transmittance

$$\mathbf{t}_{lens}(x,y) = \mathbf{p}(x,y)\exp\left[-j\frac{k}{2f}\left(x^2+y^2\right)\right], \tag{11}$$

where p(x,y) denotes a one-zero pupil function modeling the finite aperture of the lens, and where the focal distance f satisfies

$$\frac{1}{f} = \frac{1}{d_o} + \frac{1}{d_i}. \tag{12}$$

It is extremely useful, I find, for reasons clarified in the following section, to introduce the imaging property of a lens prior to discussing its Fourier transforming property. The analysis must be done with care, but it is not terribly difficult or lengthy. I proceed by assuming an object distribution of the form $U^{obj}(x,y)=\delta(x-\xi,y-\eta)$, use the Fresnel-regime propagation formula to calculate the wave amplitude incident on the lens, multiply by the lens transmittance function (with the pupil function set equal to unity—i.e., assuming an infinite aperture for the lens), and again use the Fresnel propagation formula to calculate what is the space-variant impulse response of the system,

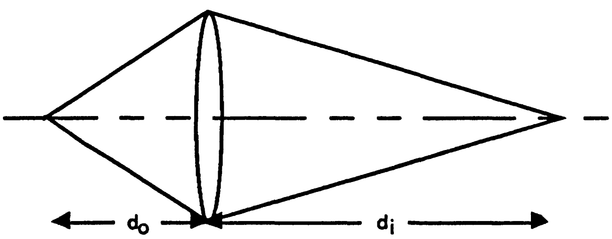

Fig. 2. Effect of thin lens is to map expanding spherical wave into converging one.

$$\mathbf{h}(x,y;\xi,\eta) = \exp\left[j\frac{k}{2d_o}\left(\xi^2+\eta^2\right)\right]\frac{1}{d_o d_i}\delta\left(\frac{\xi}{d_o}+\frac{x}{d_i},\frac{\eta}{d_o}+\frac{y}{d_i}\right)\exp\left[j\frac{k}{2d_i}\left(x^2+y^2\right)\right]. \tag{13}$$

For an arbitrary object distribution the image-plane wave amplitude is thus given by the general superposition integral

$$\mathbf{U}_{im}(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathbf{U}_{obj}(\xi,\eta)h(x,y;\xi,\eta)d\xi\, d\eta \quad . \tag{14}$$

Calculation of the integral leads to the geometrical optics image,

$$\mathbf{U}_g(x,y) = \frac{1}{M}\mathbf{U}_{obj}\left(\frac{-x}{M},\frac{-y}{M}\right), \tag{15}$$

multiplied by a quadratic phase factor, the latter being consistent with the wave field produced in the image plane if the object is simply a normally-incident plane wave.

It is important for the students to note the space-variant nature of the superposition integral: because of the (ξ,η)-dependent quadratic phase factor in the expression for $\mathbf{h}_{coh}(x,y;\xi,\eta)$, the nature of the response of the system depends on the position of the object. Although the (ξ,η) quadratic phase factor in Eq. (13) could be converted to one in (x,y), it is, I think, better in subsequent analyses that it not be.

5. FOURIER TRANSFORM PROPERTY

Goodman's treatment of the Fourier transform property of a thin spherical lens centers on the analysis of the three systems illustrated in Fig. 3. This analysis has the unfortunate effect of suggesting to the student that the Fourier transform of a wave field is found *only* in the back focal plane of the lens (many students, in fact, believe that to be the case). Better, I think, is to treat as canonical the two systems shown in Fig. 4. Analysis of the system of Fig. 4(a) is straight-forward in the Fresnel regime, with the result

$$\mathbf{U}_{FT}(x,y) = \exp\left[j\frac{k}{2d}\left(x^2+y^2\right)\right]\left[\frac{1}{\lambda d}\mathbf{T}_o\left(\frac{x}{\lambda d},\frac{y}{\lambda d}\right)\right], \tag{16}$$

where $\mathbf{T}_o(u,v)$ denotes the Fourier transform of object transmittance function $\mathbf{t}_o(x,y)$. Analysis of the system of Fig. 4(b) is less straight-forward. A brute-force analysis (performed by propagating the wave field from the point source through the object, through the lens, and to the Fourier transform plane) is *extremely* complicated, requiring the careful manipulation of multiple quadratic phase factors. Much easier is to use the inverse propagation formula to determine the *effective* (i.e., virtual in the traditional optics sense) wave field in the plane of the point source, i.e., that wave field which, if propagated forward through a distance d, would yield the actual distribution immediately beyond the object transparency.[7] This effective wave field is easily shown to have the form

$$\mathbf{U}_{eff}(x,y) = \exp\left[-j\frac{k}{2d}(x^2+y^2)\right]\left[\frac{1}{\lambda d}\mathbf{T}_o\left(\frac{-x}{\lambda d},\frac{-y}{\lambda d}\right)\right]. \tag{17}$$

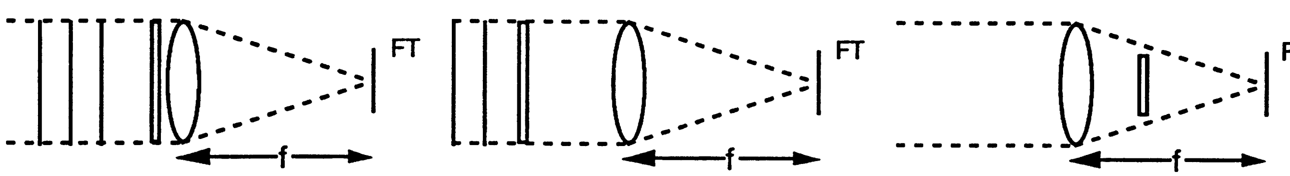

Fig. 3. Three Fourier transform cases often viewed as being canonical but not really very general.

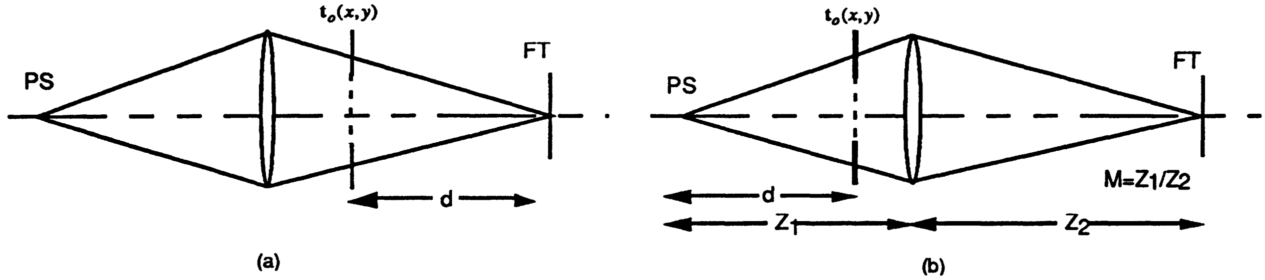

Fig. 4. Preferred canonical cases for Fourier transform operation: PS denotes a monochromatic point source, $t_o(x,y)$ the object complex amplitude transmittance, and FT the Fourier transform distribution. The distances Z_1 and Z_2 satisfy the imaging condition $1/Z_1+1/Z_2=1/f$.

The complex amplitude distribution $U_{FT}(x,y)$ in the plane conjugate to the source point is determined by calculating the image of $\mathbf{U}_{eff}(x,y)$ in that plane. Application of the imaging relationship from the previous section (which we now see to be essential to this analysis) yields, assuming no vignetting by the lens,

$$\mathbf{U}_{FT}(x,y) = \exp\left[-j\frac{k}{2d'}(x^2+y^2)\right]\left[\frac{1}{\lambda Md}\mathbf{T}_o\left(\frac{x}{\lambda Md},\frac{y}{\lambda Md}\right)\right], \tag{18}$$

the corresponding optical intensity being

$$I_{FT}(x,y) = \left(\frac{1}{\lambda Md}\right)^2\left|\mathbf{T}_o\left(\frac{x}{\lambda Md},\frac{y}{\lambda Md}\right)\right|^2. \tag{19}$$

In these expressions the factor M is given by

$$M = \frac{Z_2}{Z_1}, \tag{20}$$

which is the transverse magnification appropriate for imaging the source plane to its conjugate image plane. Not too surprisingly, the main result of the imaging operation is a magnification of the Fourier transform distribution and an inversion through the origin. The factor $1/\lambda Md$ in Eq. (18) is consistent with energy conservation for light traveling the (lossless) path from the object transparency to the Fourier transform plane. The distance d' entering into the quadratic phase factor can be determined by straight-forward manipulation of the equations. Alternatively, it can be determined with the help of Fig. 5, where the object is assumed to consist of a single on-axis pinhole. The quadratic phase factor in Eq. (18) corresponds to the spherical wave produced in the Fourier transform plane by light expanding from the pinhole. To calculate d' using the figure it is necessary to find the distance from the lens to the plane conjugate to the object plane. This distance is determined with the help of the equation

$$\frac{1}{z_1-d}+\frac{1}{z_2+d'}=\frac{1}{f}. \tag{21}$$

The distance d' can be either positive or negative, depending respectively on whether the object lies before or behind the front focal plane of the lens.

For the above analyses to be valid it is necessary that the lens be sufficiently large that its finite diameter not change the result significantly. In fact, bounds on an acceptable lens diameter have not been accurately determined, and this issue is thus somewhat open.

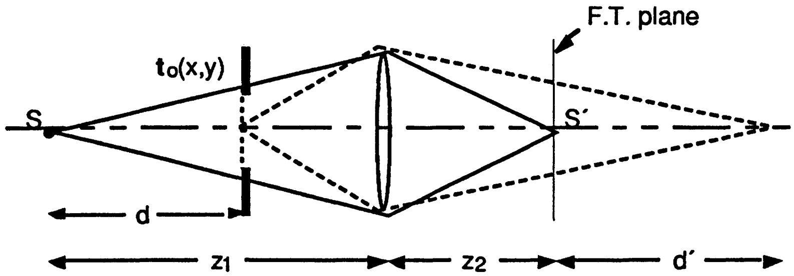

Fig. 5. Fourier transform by lens with object positioned in diverging beam. A virtual Fourier transform distribution centered on source point S is imaged by the lens to the Fourier transform (F.T.) plane centered at conjugate point S′.

A case of special interest occurs when the object is in the front focal plane of the lens.[8] In that case, the quadratic phase factor disappears and the scale factor associated with the transform is simply λf, *independent* of the distances Z_1 and Z_2:

$$\mathbf{U}_{FT}(x,y)=\frac{1}{\lambda f}\mathbf{T}_o\left(\frac{x}{\lambda f},\frac{y}{\lambda f}\right). \tag{22}$$

Also of special interest is the case when, for the system of Fig. 4(b), the source is moved to infinity. It is easily shown that in that case

$$\mathbf{U}_{FT}(x,y)=\exp\left[j\frac{k}{2f}\left(1-\frac{d}{f}\right)(x^2+y^2)\right]\left[\frac{1}{\lambda f}\mathbf{T}_o\left(\frac{x}{\lambda f},\frac{y}{\lambda f}\right)\right]. \tag{23}$$

Since the illumination of the object transparency is in this case by a normally incident plane wave, the result is equally valid for an input complex amplitude from any source. Thus, $\mathbf{t}_o$ can be replaced by an arbitrary complex amplitude $\mathbf{U}_o$ and $\mathbf{T}_o$ by the Fourier transform of $\mathbf{U}_o$.

6. COHERENT IMAGING SYSTEMS

If Goodman's text is used in teaching Fourier optics, care must be taken in the classroom discussion of coherent imaging systems, for in that text a single-lens imaging system with limiting aperture in the plane of the lens [see Fig. 6(e) below] is modeled as being space-invariant with transfer function given by a scaled pupil function, whereas in fact it can be strongly space-*variant* and should not be characterized by a transfer function at all. Indeed, Goodman's treatment of coherent imaging systems generally would lead the student to believe that most coherent imaging systems are space-invariant when in fact they are not. This failure stems partly from the absence in the text of the wave-optics analysis of a general imaging system—one characterized as in ray optics by principal planes, entrance and exit pupils, and the like—and the resulting clarification of those conditions that must be met in order for the system to be space-invariant.

I find that progressing through the analysis of a carefully chosen sequence of coherent imaging systems prepares the student for the general system case without excessive analytical difficulty. The sequence, shown in Fig. 6, consists of five special systems followed by a general system characterized by arbitrary entrance and exit pupils. The system of Fig. 6(a) is constructed of a pair of 2-f Fourier transform modules. The image, resulting from two scaled Fourier transforms performed in cascade, has the form

$$\mathbf{U}_{im}(x,y)=\mathbf{h}_{coh}(x,y)**\mathbf{U}_g(x,y), \tag{24}$$

where, as in Goodman's text, $\mathbf{U}_g(x,y)$ denotes the geometrical optics image of Eq. (15) and where the coherent point spread function $\mathbf{h}_{coh}(x,y)$ equals $(1/\lambda f_2)^2\mathbf{P}(x/\lambda f_2,y/\lambda f_2)$, $\mathbf{P}$ denoting the Fourier transform of pupil function $\mathbf{p}$. The system of Fig. 6(b), again consisting of a pair of Fourier transform modules, is described by exactly the same expressions. The system of Fig. 6(c) is obtained from that of Fig. 6(b) by combining the two lenses at the aperture plane into a single more powerful lens

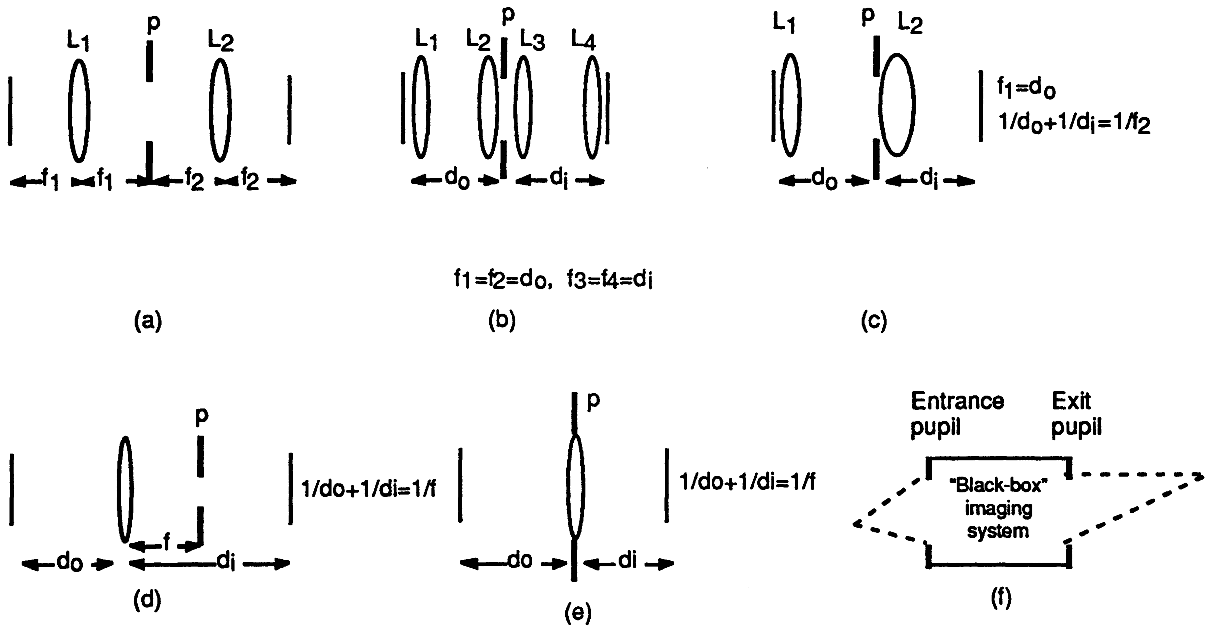

Fig. 6. Sequence of coherent imaging systems analyzed. The pupil function of the limiting aperture of each system is denoted by p.

and removing the final image-plane lens. In the ideal thin-lens regime, combining the middle lenses has no effect on system math, whereas removing the final lens introduces a quadratic phase factor. Thus, the complex amplitude of the image plane distribution is given for this system by

$$\mathbf{U}_{im}(x,y) = \left[\mathbf{U}_g(x,y) ** \mathbf{h}_{coh}(x,y)\right] \exp\left[j\frac{k}{2d_i}\left(x^2+y^2\right)\right], \tag{25}$$

where $\mathbf{U}_g$ and $\mathbf{h}_{coh}$ are essentially the same as for the two previous systems. The student should observe that removal of the first lens would prevent the Fourier transform of the object from being projected onto the limiting aperture of the system and, thus, that this first lens cannot be removed without destroying the convolutional character of the object-image relationship.

Next the system of Fig. 6(d) is analyzed. This is a single-lens system, but one for which the limiting aperture is in the back focal plane of the lens, i.e., in the plane where the Fourier transform of the object appears. The analysis is not complicated: the wave field appearing in the pupil plane is easily calculated via a Fourier-transform-by-lens formula, and a simple Fresnel-regime propagation calculation leads to the final result, an expression like that of Eq. (25) but with $(d_i\text{-}f)$ replacing d_i in the quadratic phase factor.

At this stage the single-lens imaging system of Fig. 6(e)—where the limiting aperture is *in the plane of the lens*—is considered. Although an analysis like that discussed in Sect. 4 could be used with this system, it is more informative to the student if the system of Fig. 6(c) serves as the starting point. Removal of the first lens of the system and a slight rearrangement of the second lens and aperture results in the system of Fig. 6(e), and the object-image relation is easily seen to be given not by Eq. (24) but by a modified version where $\mathbf{U}_g(x,y)$ is replaced by $(1/M)\mathbf{U}_{obj}(-x/M,-y/M)$ $\exp[j(k/2M^2d_o)(x^2+y^2)]$. The quantity $(d_i\text{-}f)$ again replaces d_i in the quadratic phase factor. The presence of the additional quadratic phase factor, which modifies the scaled object distribution prior to any convolution operation, effectively destroys any possible space invariance of the system's operation. In fact, operation of the system of Fig. 6(e) is, I think, best described by a general space-*variant* superposition integral of the form given in Eq. (14) with the impulse response now having the form

$$\mathbf{h}_{sv}(x,y;\xi,\eta)=\exp\left[j\frac{k}{2d_o}(\xi^2+\eta^2)\right]\frac{1}{\lambda^2 d_o d_i}\mathbf{P}\left(\frac{x}{\lambda d_i}+\frac{\xi}{\lambda d_o},\frac{y}{\lambda d_i}+\frac{\eta}{\lambda d_o}\right)\exp\left[j\frac{k}{2d_i}(x^2+y^2)\right]. \quad (26)$$

This particular form of object-image relationship has a satisfyingly symmetric nature: geometrical magnification is seen to be a consequence of the way the delta-function is scaled in object- and image-space coordinates, the space-variance is seen to be a consequence (through the first quadratic phase factor) of the distance of object point (ξ,η) from the lens aperture, and resolution-limiting blur can be viewed as operating on either the object or the image distribution.

Analysis of a general system, as illustrated in Fig. 6(f), follows fairly naturally from that of the system of Fig. 6(e). Details of such an analysis, which would lengthen this paper overmuch, are presented in Ref. 9. The system, which is in general space-variant, is described by the space-variant superposition integral of Eq. (14) with impulse response given by

$$\mathbf{h}_{sv}(x,y;\xi,\eta)=\exp\left[j\frac{k}{2R_1}(\xi^2+\eta^2)\right]\frac{1}{\lambda^2 d_o d_i}\mathbf{P}_s\left(\frac{\xi}{\lambda d_o}+\frac{x}{\lambda d_i},\frac{\eta}{\lambda d_o}+\frac{y}{\lambda d_i}\right)\times\exp\left[j\frac{k}{2R_2}(x^2+y^2)\right], \quad (27)$$

where the parameters R_1, R_2, d_o, and d_i, along with the Fourier transform $\mathbf{P_S}$ of a system pupil function $\mathbf{p_S}(x,)$, are easily found by means of a ray-optics-based analysis of the system. (Parameters R_1 and R_2 are, respectively, the distance from the object to the entrance pupil and the distance from the exit pupil to the image. In the case of the system of Fig. 6(e) $R_1 = d_o$ and $R_2 = d_i$. Only when $R_1 = \infty$, i.e., when the entrance pupil is at infinity, can the space variance be removed and a convolutional expression of the form in Eq. (25) be used.

7. SUMMARY OF IMPORTANT POINTS

Major points raised in this paper are worth repeating. In the teaching of a course on Fourier optics, I urge that (a) the angular spectrum approach be employed in the analysis of wave propagation/diffraction; (b) inverse propagation be introduced as a tool; (c) the canonical Fourier transform configurations of Fig. 4 rather than those of Fig. 3 be analyzed; (d) the correct space-variant analysis be applied to a single-lens imaging system where the limiting aperture is at the lens rather than in its back focal plane; and (e) the case of a general system for coherent imaging be treated and that it be made clear that—unlike the case for incoherent imaging—*most* coherent imaging systems are in fact space-variant.

8. FOOTNOTES AND REFERENCES

1. Fourier optics is also strongly identified with the study of linear and quadratic phase factors in the analysis of optical phenomena and systems.
2. W. T. Rhodes, Introduction to Fourier Optics and Optical Signal Processing, in preparation.
3. J. W. Goodman, Introduction to Fourier Optics, 2nd Ed., McGraw-Hill Book Company, New York 1996.
4. J. D. Gaskill, Linear Systems, Fourier Transforms, and Optics, John Wiley and Sons, New York 1978.
5. Students still need to be introduced to the Huygens-Fresnel principle, but this can be done when they are introduced to the Fresnel-regime propagation integral.
6. See, e.g., H. Stark, "Theory and Measurement of the Optical Fourier Transform," in H. Stark, ed., Applications of Optical Fourier Transforms, Academic, New York, 1982, pp. 2-7.
7. I am indebted to P. Andres of the University of Valencia for making me realize the significance of this trick and to V. Morozov for showing me additional applications.
8. This special case is noted by Gaskill.
9. William T. Rhodes, "Simple procedure for the analysis of coherent imaging systems," Opt. Let. 19(19), 1559-1561 (1994).